Combinatorics and Graph Theory I, Spring 2019, Sheet 3

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, March 14th.

Exercise 3.1
Sketch the network whose vertices are s, a, b, c, d, t; and whose arcs and capacities are

$(x,y):$	(s,a)	(s,b)	(a,b)	(a,c)	(b,d)	(d,c)	(c,t)	(d,t)
$c(x,y):$	5	3	3	3	5	2	6	2

Find a flow with value 7 and a cut with capacity 7. What is the value of the maximum flow, and why?

Exercise 3.2
Let $\overrightarrow{G} = (V; \overrightarrow{E})$ be a digraph with source s and sink t and suppose $\phi : \overrightarrow{E} \to \mathbb{R}$ be a function, not necessarily a flow.

(a) Show that

$$\sum_{x \in V} \phi(\{x\}, V \setminus \{x\}) = \sum_{x \in V} \phi(V \setminus \{x\}, \{x\}).$$

(b) Deduce from (a) that if ϕ is a flow then the net flow out of s is equal to the net flow into t:

$$\phi(\{s\}, V \setminus \{s\}) - \phi(V \setminus \{s\}, \{s\}) = \phi(V \setminus \{t\}, \{t\}) - \phi(\{t\}, V \setminus \{t\})$$

Exercise 3.3
Suppose $S \subseteq \overrightarrow{E}$ is a set of edges after whose deletion there is no flow from s to t with strictly positive value. Prove that S contains a cut separating s from t, i.e., there is $X \subset V$ with $s \in X$ and $t \notin X$ such that $\overrightarrow{E}(X, V \setminus X) \subset S$.

Exercise 3.4
Let $f : \overrightarrow{E} \to \mathbb{R}^+$ be a flow on a digraph $\overrightarrow{G} = (V; \overrightarrow{E})$ with source s, sink t and capacity function $c : \overrightarrow{E} \to \mathbb{R}^+$.

(a) Define the *value* of the flow f.

(b) Suppose that $X \subset V$ contains s but not t. Show that the value of f is also equal to

$$f(X, V \setminus X) - f(V \setminus X, X)$$

(c) Using (b) and the fact that $0 \leq f(x,y) \leq c(x,y)$ for each $(x,y) \in \overrightarrow{E}$, prove that the value of f is at most equal to the capacity of a cut $\overrightarrow{E}(X, V \setminus X)$ separating s from t.

Notations and reference

Please refer to Chapter III (Flows, Connectivity and Matching), Section 1 (Flows in Directed Graphs) in Bollobás Modern Graph Theory, 1998 for the relevant material.

In particular, for this exercise sheet, for a digraph $\overrightarrow{G} = (V; \overrightarrow{E})$ and a function $\phi : \overrightarrow{E} \to \mathbb{R}$, for $X, Y \subseteq V$, we define

$$\overrightarrow{E}(X,Y) := \{(x,y) \in \overrightarrow{E} \mid x \in X \text{ and } y \in Y\},$$

and

$$\phi(X,Y) := \sum_{x \in X, y \in Y, (x,y) \in \overrightarrow{E}} \phi(x,y).$$