Combinatorics and Graph Theory I, Spring 2019, Sheet 4

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, March 21st.

Exercise 4.1
$\overrightarrow{G} = (V; \overrightarrow{E})$ be a digraph with source s and sink t and suppose $f_1, f_2 : \overrightarrow{E} \to \mathbb{R}$ be two flows. The *flow sum* $f_1 + f_2$ is the function from $\overrightarrow{E}$ to $\mathbb{R}$ defined by

$$(f_1 + f_2)(u, v) = f_1(u, v) + f_2(u, v)$$

for all $(u, v) \in \overrightarrow{E}$. If f_1 and f_2 are flows in $\overrightarrow{G}$, which of the flow properties must the flow sum $f_1 + f_2$ satisfy, and which might it violate?

Exercise 4.2
Let $\overrightarrow{G} = (V; \overrightarrow{E})$ be a digraph with source s and sink t and suppose $f : \overrightarrow{E} \to \mathbb{R}$ be a flow. Let $\alpha \in \mathbb{R}$ and define *scalar flow product* $(\alpha f) : \overrightarrow{E} \to \mathbb{R}$ as

$$(\alpha f)(u, v) = \alpha \cdot f(u, v)$$

for all $(u, v) \in \overrightarrow{E}$. Prove that the flows in a network form a *convex set*. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all $0 \le \alpha \le 1$.

Exercise 4.3
Decide whether each of the following statements is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

(a) Let $\overrightarrow{G} = (V; \overrightarrow{E})$ be a digraph with source s and sink t, and a positive integer capacity $c(e)$ on every edge e. If f is a maximum $s - t$ flow in $\overrightarrow{G}$, then f saturates every edge out of s with flow (i.e., for all edges e out of s, we have $f(e) = c(e)$).

(b) Let $\overrightarrow{G} = (V; \overrightarrow{E})$ be a digraph with source s and sink t, and a positive integer capacity $c(e)$ on every edge e. Let S be a minimum s-t cut with respect to these capacities $\{c(e) \mid e \in \overrightarrow{E}\}$. Now suppose we add 1 to every capacity; then S is still a minimum s-t cut with respect to these new capacities $\{c(e) + 1 \mid e \in \overrightarrow{E}\}$

Exercise 4.4
Consider a set of mobile computing clients in a certain town who each need to be connected to one of several possible base stations. We'll suppose there are n clients, with the position of each client specified by its (x, y) coordinates in the plane. There are also k base stations; the position of each of these is specified by (x, y) coordinates as well.

For each client, we wish to connect it to exactly one of the base stations. Our choice of connections is constrained in the following ways. There is a range parameter r: a client can only be connected to a base station that is within distance r. There is also a load parameter L: no more than L clients can be connected to any single base station.

Your goal is to model the following problem as a flow problem. Given the positions of a set of clients and a set of base stations, as well as the range and load parameters, decide whether every client can be connected simultaneously to a base station, subject to the range and load conditions in the previous paragraph. This means that the decision to the flow problem should be yes (which would mean there exists a flow of a certain value) if and only if the decision to the connection problem described above is yes. Explain why the modelling is correct.

Exercise 4.5

Statistically, the arrival of spring typically results in increased accidents and increased need for emergency medical treatment, which often requires blood transfusions. Consider the problem faced by a hospital that is trying to evaluate whether its blood supply is sufficient. The basic rule for blood donation is the following. A person's own blood supply has certain *antigens* present (we can think of antigens as a kind of molecular signature); and a person cannot receive blood with a particular antigen if their own blood does not have this antigen present. Concretely, this principle underpins the division of blood into four types: A, B, AB, and O. Blood of type A has the A antigen, blood of type B has the B antigen, blood of type AB has both, and blood of type O has neither. Thus, patients with type A can receive only blood types A or O in a transfusion, patients with type B can receive only B or O, patients with type O can receive only O, and patients with type AB can receive any of the four types.

(a) Let s_O, s_A, s_B, and s_{AB} denote the supply in whole units of the different blood types on hand. Assume that the hospital knows the projected demand for each blood type d_O, d_A, d_B, and d_{AB} for the coming week. Model the task of evaluating if the blood on hand would suffice for the projected need as a flow problem. Explain why your model is correct.

(b) Consider the following example. Over the next week, they expect to need at most 100 units of blood. The typical distribution of blood types is roughly 45 percent type O, 42 percent type A, 10 percent type B, and 3 percent type AB. The hospital wants to know if the blood supply it has on hand would be enough if 100 patients arrive with the expected type distribution. There is a total of 105 units of blood on hand. The table below gives these demands, and the supply on hand.

blood type	supply	demand
O	50	45
A	36	42
B	11	8
AB	8	3

Is the 105 units of blood on hand enough to satisfy the 100 units of demand? Find an allocation that satisfies the maximum possible number of patients. Use an argument based on a minimum-capacity cut to show why not all patients can receive blood. Also, provide an explanation for this fact that would be understandable to the clinic administrators, who have not taken a course on graph theory. (So, for example, this explanation should not involve the words flow, cut, or graph.)