Combinatorics and Graph Theory I, Spring 2019, Sheet 5

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, April 4th.

Notations: All graphs in this sheet are undirected. For an undirected graph $G = (V, E)$, neighbourhood of a vertex u is defined as $N_G(u) := \{v \mid \{u, v\} \in E\}$. For $S \subseteq V$, neighbourhood of S is defined as $N(S) := (\cup_{v \in S} N(v)) \setminus S$. For $S \subseteq V$, $G - S$ is obtained by deleting S and all edges incident on S. More formally, the $G - S := (V \setminus S, E \setminus E')$, where $E' = \{\{u, v\} \mid u, v \in S\}$.

Exercise 5.1

(a) Design a bipartite graph $G = (A \uplus B, E)$ with $|A| = |B| = 5$, no isolated vertices, and at least 10 edges, in which a *maximum* matching M has size 4, while there exists a *maximal* matching M' of strictly smaller size. Identify M and M' in your graph.

(b) Identify a set $S \subseteq A$ for which $|N(S)| < S$ (observe that $|N(A)| = |A|$ as there are no isolated vertices). Why such a set S must exist?

Exercise 5.2

An augmenting path with respect to a matching M in a bipartite graph G is a path P in G which starts at an unmatched vertex and then contains alternately an edge not in M and an edge in M and ends in another unmatched vertex. The symmetric difference of the edges of P and the edges of M is again a matching and covers two more vertices than M (the endpoints of P).

a) Let M be a matching in a bipartite graph G. Show that if M contains fewer edges than some other matching N in G then G contains an augmenting path with respect to M.

b) Find an augmenting path P in your graph of the first exercise with respect to the matching M'.

c) Using P, construct a matching of size larger than M'. Repeat the process if the current matching does not have size 4 to get to a matching of size 4.

Exercise 5.3

In a graph $G = (V, E)$, a minimal a-b separator for is an inclusion-wise minimal set $S \subseteq V \setminus \{a, b\}$ such that there is no path from a to b in $G - S$. Let C_a and C_b be the connected components of $G - S$ containing a and b respectively. Show that $N(C_a) = N(C_b) = S$, that is, each vertex in S has a neighbour in both C_a and C_b.