Combinatorics and Graph Theory I, Spring 2019, Sheet 6

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, April 11th.

Exercise 6.1
Menger's theorem states that a graph $G = (V, E)$ is k-vertex connected if and only if for any two vertices $u, v \in V$, there exist k vertex disjoint paths between u and v. The "only if" direction was proved in the lecture. Prove the "if" direction. Do the same for the edge version of Menger's theorem, i.e., prove that if there are k edge-disjoint paths between any two vertices $u, v \in V$, then G is k-edge connected.

Exercise 6.2
Use Menger's theorem to prove that a graph $G = (V, E)$ is 2-vertex connected if and only if for any two vertices $u, v \in V$, there exists a cycle containing both u and v.

Exercise 6.3
Let $G = (V, E)$ be a k-vertex connected graph. Show that for every $v \in V$ and $S \subseteq V \setminus \{v\}$ such that $|S| \geq k$, there are k vertex disjoint paths $P_1, P_2, \ldots, P_k$ such that each P_i starts at v, ends at a vertex of S, and contains no other vertices of S.
(Hint: Add a new vertex v adjacent to every vertex in S to get a new graph G'. Argue about the vertex-connectivity of G' and apply Menger's theorem.)

Exercise 6.4
Using the result in Exercise 6.3, show that for $k \geq 2$, for a k-vertex connected graph $G = (V, E)$, for any $S \subseteq V$ such that $|S| = k$, there is a cycle that contains all vertices of S. Is the converse true, like in Exercise 6.2?
(Hint: Look at the cycle C containing largest number of vertices from S. Apply previous exercise to C and a vertex of S not in C. Be careful about the case when C has less than k vertices. For that, use the fact that every k-vertex connected graph is also k'-veretx connected for all $k' \leq k$.)

Exercise 6.5
A k-vertex connected graph $G = (V, E)$ is critically k-connected if for any $e \in E$, $G - e := (V, E \setminus \{e\})$ is not k-vertex connected.

(a) For a critically 2-connected graph G, show that G has a vertex of degree 2.

(b) For each n, find an example of a critically 2-connected graph with a vertex of degree at least n.

(Hint: Use the definition of ear decompositions for 2-connected graphs taught in the lecture.)