Combinatorics and Graph Theory I, Spring 2019, Sheet 9

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, May 2nd.

Exercise 9.1
Two finite set systems $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are isomorphic if there exists a bijection $f : X \to X'$ such that for any $S \subseteq X$, $S \in \mathcal{L}$ if and only if $(\cup_{x \in S} f(x)) \in \mathcal{L}'$. Show that Fano plane is the only projective plane of order 2, upto isomorphism, i.e., any projective plane of order 2 is isomorphic to it.

Exercise 9.2
Let X be a fnite set and $\mathcal{L}$ a system of lines (subsets of X) satisfying conditions (P1) and (P2), and the following condition:

(P0'): There exist at least two distinct lines having at least three points each.

Prove that any such $(X, \mathcal{L})$ is a finite projective plane. (Show that the set system satisfies the condition (P0) taught in the lecture).

Exercise 9.3

(a) Find an example of a set system $(X, \mathcal{L})$ on a nonempty finite set X that satisfies conditions (P1) and (P2) but does not satisfy (P0).

(b) Find an X and $\mathcal{L}$ as in (a) such that $|X| \geq 10$, $\mathcal{L} \geq 10$ and each $L \in \mathcal{L}$ has at least 2 points.

(c) Describe all set systems $(X, \mathcal{L})$ as in (a).

Exercise 9.4
Let $(X, \mathcal{L})$ be a finite projective plane with set of points X and set of lines $\mathcal{L}$. Let r be the order of $(X, \mathcal{L})$, defined as the number of points minus one in any given line, i.e., $r = ||\mathcal{L}| - 1$ for all $L \in \mathcal{L}$. The incidence graph (or *Levi graph*) of $(X, \mathcal{L})$ is the bipartite graph on $X \cup \mathcal{L}$ with an edge joining $x \in X$ to $L \in \mathcal{L}$ precisely when $x \in L$.

(a) The girth of a graph with at least one cycle is the smallest positive integer g for which there is a g-cycle. (Thus for instance a triangle-free graph has girth at least 4.) A k-regular graph is a graph in which each vertex has degree k.

Show that a k-regular graph with girth $2m+1$ must have at least $1 + k + k(k-1) + \cdots + k(k-1)^{m-1}$ vertices, and that a k-regular graph with girth $2m$ must have at least $2[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{m-1}]$ vertices.

(b) Show that the incidence graph of $(X, \mathcal{L})$ is an $(r+1)$-regular graph of girth 6 which attains the lower bound given in (a) for $m = 3$. (This would imply that the incidence graph of a projective plane of order r has the minimum number of vertices among all $(r+1)$-regular graphs of girth 6.)