Advanced Graph Algorithms Jan-Apr 2014

Lecture 24 — April 23, 2014

Lecturer: Sanjukta Roy *Scribe: Sanjukta Roy*

1 Overview

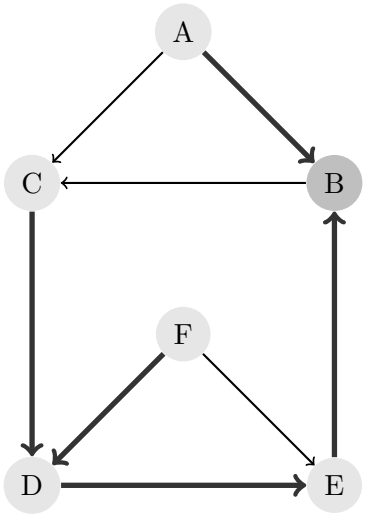
In the last presentation the lemma of Gessel-Viennot was presented. Today we will see the proof of Tutte's Matrix Tree Theorem using the lemma. A classical theorem of graph theory, due essentially to Kirchhoff, relates the number of spanning trees of an undirected connected graph to the determinant of a certain matrix. We discuss an important generalization to directed graphs, first established by W. T. Tutte, and give the proof. The proof was suggested by Jürgen Schütz.

2 Some Definitions

Definition 1. *Arborescences in Directed Graphs.*
Suppose $T = (V, E)$ is a directed graph and u a vertex. We say that T is an arborescence converging to u if there is a directed path from any $v \neq u$ to u, and the out-degree of v in T, denoted by $d_T^-(v)$ equals 1 for all $v \neq u$ (and $d_T^-(u) = 0$). This implies, of course, that there is exactly one such path from any v to u.

Definition 2. *Spanning Arborescence.*
Given a directed graph $G = (V , E)$, then a subgraph $T \subseteq G$ is called a spanning arborescence converging to u if T is an arborescence towards u, and $V (T) = V (G)$.
Similarly, $T \subseteq G$ is a spanning arborescence diverging from u if there is (precisely) one directed path in T from u to any vertex $v \neq u$ and in-degree of v in T, denoted as $d_T^+(v) = 1$ for $v \neq u, d_T^+(u) = 0$.

Example: In the figure below, the bold edges show a spanning arborescence converging to B.



Definition 3. *Laplace Matrix.*
Let $G = (V , E)$ be a directed loopless graph with $V = \{1, \ldots, n\}$, and denote by $A = (a_{ij})$ the $n \times n$-matrix with

$$a_{ij} = \textit{number of directed edges } i \to j.$$

Furthermore, let D^- and D^+ be diagonal matrices, where $(i, i)^{th}$ entry is outdegree and indegree of i^{th} vertex respectively, rest of the entries are 0.

$$D^- = \begin{pmatrix} d^-(1) & & & 0 \\ & d^-(2) & & \\ & & \ddots & \\ 0 & & & d^-(n) \end{pmatrix}$$

$$D^+ = \begin{pmatrix} d^+(1) & & & 0 \\ & d^+(2) & & \\ & & \ddots & \\ 0 & & & d^+(n) \end{pmatrix}$$

The Laplace matrices $L^-(G)$ and $L^+(G)$ are then
$L^-(G) = D^- - A$ and $L^+(G) = D^+ - A$.
Note, for any matrix $M = (m_{ij})$ we denote as usual by $M_{s,t}$ the (n 1) x (n 1)-submatrix where row s and column t have been deleted. Now we are all set for the theorem of Tutte.

3 Goal

Let, $t^-(G, u)$ and $t^+(G, u)$ be the number of spanning arborescences in G converging to u and diverging from u, respectively for a given graph G = (V , E).
Our goal is to compute $t^-(G, u)$ and $t^+(G, u)$.

Theorem 4 (Tutte's Matrix Tree Throrem)**.** *Let $G = (V , E)$ be a loopless directed graph, $V = \{1, \ldots, n\}$. Then for any $s, r \in V$ (where $s = r$ is possible),*

$$t^-(G, s) = (1)^{s+r} det L^-(G)_{s,r} \tag{1}$$

$$t^+(G, s) = (1)^{s+r} det L^+(G)_{r,s} \tag{2}$$

4 Proof of Tutte's Matrix Tree Throrem

Proof. We first give a proof of the (1) using the lemma of Gessel-Viennot then show a equivalence between (1) and (2), which completes the proof. Since loops clearly play no role, we may and will assume that G contains no loops, but there may be multiple edges between $v \neq w$ in either direction.

Recall,

Lemma 5. *The Lemma of Gessel-Viennot.*
Let G = (V , E) be a directed acyclic graph with weights on E, $A = \{A_1, \ldots, A_n\}, B = \{B_1, \ldots, B_n\}$, and M the pathmatrix from A to B. Then

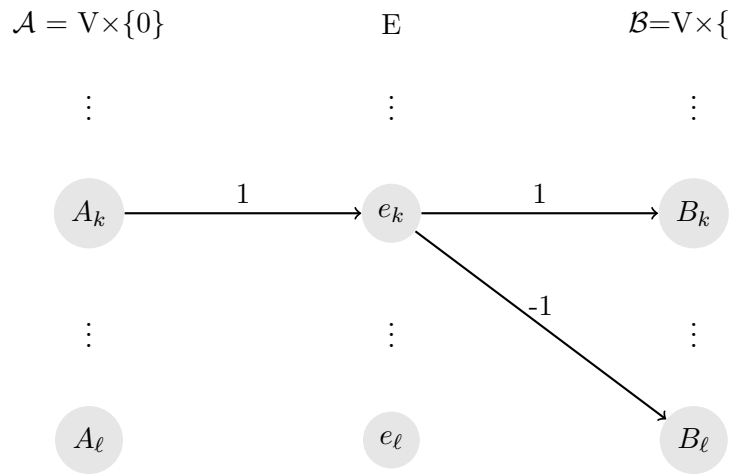
$$detM = \sum_{\mathcal{P} \in V.D} (Sgn\mathcal{P}).w(\mathcal{P}) \tag{3}$$

where V.D denotes the set of vertex disjint path system, $(Sgn\mathcal{P})$ is the sign of the permutation associated with the path system $\mathcal{P}$ and $w(\mathcal{P})$ is the product of weights of the paths in the path system. And the path -matrix $M = (m_{ij})$ is given by

$$m_{ij} = \sum_{\mathcal{P}: A_i \to B_j} w(\mathcal{P}) \tag{4}$$

Let the directed graph G = (V , E) be given. In order to prove (1) we construct a weighted directed graph H and two vertex sets A and B of H whose path matrix will be precisely $L^-(G)$. It remains then to find a correspondence between the vertex-disjoint path systems of H and the arborescences in G.
The graph H is declared as follows. The vertex set is $(V \times \{0\}) \biguplus (V \times \{1\}) \biguplus E$, where we use the notation $A_k = (k, 0)$ and $B_k = (k, 1)$ for $k = 1, \ldots, n$. To every edge $e = (k, l) \in E$ we have the H-edges $A_k \to e, e \to B_k$, each of weight 1, and further $e \to B_l$ of weight -1.

The paths of the form $A_k \xrightarrow{1} e \xrightarrow{1} B_k$ adds total weight of $d^-(k)$ to the path system for $k = \{1, \ldots, n\}$ (using equation (4)). Similarly, paths of the form $A_k \xrightarrow{1} e \xrightarrow{-1} B_l$ for $k \neq l$ contributes a value of $-a_{kl}$ to the path system for all $(k, l) \in E$.
Therefore, from (4) the path-matrix with respect to $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ is $D^- - A$ which is exactly $L^-(G)$.



Edges in H corresponding to $(k, \ell) \in E$.

We want to find the number of spanning arborescence converging to s. We arbitarily fix a vertex r and make the following changes to H.
Delete all edges incident with A_s and B_r and add a single edge $A_s \to B_r$ (or no edge if s=r) of

weight 1. Call this new graph H'.

Let the path-matrix for H' be M. Note M is same as $L^-(G)$ except the entries in the s^{th} row and r^{th} column are 0 but the $(s,r)^{th}$ entry is 1. Using row and column exchanges,

$$detM = (-1)^{(s+r)}.detL^-(G)_{s,r}$$

which is the right-hand side of (1).

We want to use (3) to prove (1), so now we look at the vertex disjoint path systems in H'. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be such a system, $P_i : A_i B_{\sigma(i)}$. The paths are of the form

$$P_s : A_s \to B_{\sigma(s)} = B_r, w(P_s) = 1,$$
$$P_k : A_k \to e_k \to B_{\sigma(k)}, k \neq s, \sigma(k) = k, w(P_k) = 1$$
$$P_k : A_k \to e_k \to B_{\sigma(k)}, k \neq s, \sigma(k) \neq k, w(P_k) = -1$$

where k is the initial vertex of $e_k \in E$.
Let $\sigma = \sigma_0\sigma_1 \ldots \sigma_h$ be the cycle decomposition of σ in which we have omitted the fixed points (other than s, if there is any), and σ_0 is the cycle containing $s, \sigma_0(s) = t$.
Look at a cycle $\sigma_i (i \geq 1)$ of, say, length l ; then
$A_k \to e_k \to B_{\sigma(k)}, A_{\sigma(k)} \to e_{\sigma(k)} \to B_{\sigma^2(k)}, \ldots, A_{\sigma^{l-1}(k)} \to e_{\sigma^{l-1}(k)} \to B_k$.
Since we are not looking at fixed points, all these paths are of third kind, hence have weight -1, and so

$$w(P_k)w(P_{\sigma(k)}) \ldots w(P_{\sigma^{l-1}(k)}) = (-1)^l$$
$$sgn(\sigma_i) = (-1)^{l-1},$$

since there are $(l-1)$ transpositions.

$$sgn(\sigma_i).w(P_k)w(P_{\sigma(k)}) \ldots w(P_{\sigma^{l-1}(k)}) = (-1)^{l-1}.(-1)^l = -1$$

Similarly for σ_0 , we obtain

$$(sgn\sigma_0)w(P_s)w(P_{\sigma(s)}) \ldots w(P_{\sigma^{l-1}(s)}) = (-1)^{l-1}.(-1)^{l-1} = 1$$

since $w(P_s) = 1$, while the other paths have weight -1.
In summary, we see that

$$(signP)w(P) = (-1)^h, \tag{5}$$

with h as defined above, $\sigma = \sigma_0\sigma_1 \ldots \sigma_h$.

Let us define the subgraph $T_P = (V, \{e_k : k \neq s, k = 1, \ldots, n\})$ of G corresponding to each vertex-disjoint path system $\mathcal{P}$. Since the paths in $\mathcal{P}$ are vertex-disjoint we obtain

$$d^-_{T_P}(k) = \begin{cases} 1 & k \neq s, \\ 0 & k = s. \end{cases} \tag{6}$$

The following two assertions will finish the proof.

Lemma 6. *If T is a spanning arborescence in G converging to s, then there is precisely one vertex-disjoint path system in H' with $T_P = T$. Furthermore, $(sgn\mathcal{P})w(\mathcal{P}) = 1$.*

Lemma 7. *Call the set of all vertex-disjoint path systems $\mathcal{P}$ for which T_P is not an arborescence converging to u, $\mathcal{P}_{bad}$. There is an involution $\varphi : \mathcal{P}_{bad} \to \mathcal{P}_{bad}$*

$$(sgn\varphi\mathcal{P})w(\varphi\mathcal{P}) = -(sgn\mathcal{P})w(\mathcal{P})$$

where $\mathcal{P} \in \mathcal{P}_{bad}$

If we assume the above two statements are correct, the lemma of GesselViennot will then give

$$\begin{aligned}(-1)^{s+r} detL^-(G)_{s,r} &= \sum_{\mathcal{P}\in V.D} (sgn\mathcal{P})w(\mathcal{P}) \\ &= \sum_{T_P} (sgn\mathcal{P})w(\mathcal{P}), \\ &= t^-(G,s),\end{aligned}$$

where T_P is arborescence to s.

Hence it proves (1).

Proof of Lemma 6. Let T be a spanning arborescence in G converging to s, and denoted by e_k, the unique edge leaving $k \neq s$ in T . We construct of a path system $\mathcal{P}$ such that the subgraph corresponding to $\mathcal{P}, T_P$ is T.
Suppose, Q is $r = k_1 \to \ldots \to k_l = s$, the path from r to s. The system $\mathcal{P}$ in H' is,
$P_s : A_s \to B_r$,
$P_{k_i} : A_{k_i} \to e_{k_i} B_{k_{i+1}} (i = 1, \ldots, l)$,
and $P_k : A_k \to e_k \to B_k$(k not on Q),

Clearly $\mathcal{P}$ is vertex-disjoint with $T_P = T$. Since all cycles not containing s are fixed points

$$(sgn\mathcal{P})w(\mathcal{P}) = 1 \text{ [by (5)]}.$$

Since T is spanning arborescence, there is only one edge leaving each vertex except s, Therefore for a given tree T we can construct only a unique path system.

□

Proof of Lemma 7. If T_P is not an arborescence toward s, then by (6) since each vertex has out-degree atmost 1 and there are n vertices and n-1 edges T_P must contain directed cycles.
Let k be minimal such that k is on a cycle C in T_P , $C = i_1 \xrightarrow{e_{i_1}} i_2 \xrightarrow{e_{i_2}} \ldots \to i_l \xrightarrow{e_{i_l}} i_1$.
Note that $k \notin \sigma_0$. For the paths P_{i_j} in $\mathcal{P}$ there are two possibilities:

$$P_{i_j} : A_{i_j} \to e_{i_j} \to B_{i_j} \text{ for all } j = 1, \ldots, l, or$$
$$P_{i_j} : A_{i_j} \to e_{i_j} \to B_{i_j+1} \text{ for all } j = 1, \ldots, l.$$

Denote by $\varphi(\mathcal{P})$ the system in which the paths P_{i_j} follow the other possibility. Since $T_{\varphi(\mathcal{P})} = T_{\mathcal{P}}$, hence φ is an involution. Now we observe that φ changes the number h in $\sigma = \sigma_0\sigma_1 \ldots \sigma_h$ precisely by 1, and we conclude from (5) that

$$(sgn\varphi\mathcal{P})w(\varphi\mathcal{P}) = -(sgn\mathcal{P})w(\mathcal{P})$$

.

□

Claim 8. *(1) and (2) are Equivalent*

Proof. Let G' be the graph derived from G by reversing all directions. Then

$$t^+(G,s) = t^-(G',s), L^+(G) = L^-(G')^T$$

, and therefore for $s, r \in V$,

$$t^+(G,s) = t^-(G',s) = (-1)^{s+r} detL^-(G')_{s,r}$$

$$= (-1)^{s+r} detL^+(G)_{r,s},$$

as claimed. It thus suffices to prove (2) for converging arbores- cences. □

And thus we have proved the theorem.

□

4.1 Convergence to Kirchhoff's result

Let t(G) be the number of spanning trees of an undirected graph G. Replace each edge v, w of G by a pair of directed edges $(v,w),(w,v)$ in each direction, and call the resulting graph G' .

Claim 9. *For any $u \in V$,*

$$t(G) = t^-(G',u) = t^+(G',u).$$

Proof. Note that each spanning arborescence in G' converging to u (or diverging from u) yields a spanning tree in G, by deleting the orientations.
Conversely, let T be a spanning tree of G. Orienting the edges of T in the (unique) way toward u (respectively away from u) gives a spanning arborescence in G'. This is unique because T is a tree and there is only one path from each vertex to u (to each vertex from u).Therefore it is a bijection. □

4.2 Corollary

Corollary 10. *In an Eulerian directed graph G, there is the same num- ber of arborescences converging to a vertex u (or diverging from u), for all $u \in V$.*
That is, $d^-(i) = d^+(i)$ for all $i \in V$, hence $L^-(G) = L^+(G)$. Then for $s, r \in V$, we conclude from (1) that

$$t^-(G,s) = (-1)^{s+r} detL^-(G)_{s,r}$$

$$= (-1)^{s+r} det L^+(G)_{s,r}$$
$$= t^+(G, r).$$

Corollary 11. *Let G = (V , E) be a loopless undirected graph, A the adjacency matrix, and D the diagonal matrix of the vertex-degrees, L(G) = D - A. Then*

$$t(G) = (-1)^{s+r} L(G)_{s,r}$$

for any $s, r \in V$.

References

[1] I. Gessel and G. Viennot (1985): *Binomial determinants, paths, and hook length formulae.* Advances Math. 58, 300321.

[2] W.T. Tutte (1984): *Graph Theory.* AddisonWesley, Reading.

[3] M. Aigner (2000): *A Course in Enumeration*, ISBN 978-3-540-39032-4 Springer Berlin Heidelberg New York