Advanced Graph Algorithms Jan-Apr 2014

Lecture 3 — January 10, 2014

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1 Overview

In the last lecture, we have seen the the power of dynamic programming in trees, and defined classes of graphs which have some tree-like structure. We hoped that these would enable us to use dynamic programming on them to solve problems for these graphs, that are NP-Complete for general graphs.

To do this, we defined the idea of a graph being $t-$decomposable if, for every sub-graph of the graph, we can find, within it, a set of vertices of size at most t such that removal of those vertices from the sub-graph, leaves the remaining connected components of size at most half the size of the original sub-graph. This helped us to 'decompose' the graph in a recursive way. But, we saw that it was not a proper-tree like structure because we could not prevent edges from vertices in nodes of the tree to vertices in non-immediate ancestors. This property was crucial for carrying out dynamic programing for the Maximum Independent Set problem.

In this lecture we will first define *Weighted Balanced Separators* and $W-$*balanced Separators*. Then we shall use them to enrich our idea of $t-$decomposability of a graph and use it to show the existence of a desirable decomposition that is tree-like.

2 Main Section

2.1 Some Preliminary Definitions

Definition 1 (Weighted Balanced Separator)**.** *For a function $w : 2^{V(G)} \to \mathbb{R}_{\geq 0}$ which follows the property that for any $S \subseteq V(G)$, $w(S) = \Sigma_{v \in S} w(\{v\})$, a Weighted Balanced Separator of a graph G is a set $X \subseteq V(G)$ such that for any connected component C of $G \setminus X$, $w(V(C)) \leq \frac{w(V(G))}{2}$.*

Lemma 2. *A tree has a Weighted Balanced Separator of size 1.*

Proof. Without loss of generality, assume the tree is rooted at r (if it isn't, pick a vertex arbitrarily). For a vertex v, let T_v be the graph induced on v and all it's decendants. We now give an algorithm to find such a vertex:

Step 1: Let the root be r and let it's children be $r_1, r_2, \ldots, r_l$.

Step 2: Check if r is the required vertex. If so, stop and return r.

Step 3: If not, it is because there is some $i \in \{1, \ldots, l\}$ such that $w(V(T_{r_i})) > \frac{w(V(G))}{2}$. Go to *Step 1* taking r_i as the tree.

Clearly, this algorithm terminates because the tree is of finite height. Let $T_1, T_2, \ldots, T_l$ be the components of $T \setminus \{v\}$. If (without loss of generality) $w(V(T_l)) > \frac{w(V(T))}{2}$, let the root of T_l be r_l. Then $T_1 \cup T_2 \cup \ldots \cup T_{l-1} \cup \{r\}$ is a component of $T \setminus \{r_l\}$ and further, $w(V(T_1)) + w(V(T_2)) + \ldots + w(V(T_{l-1})) + 1 \leq \frac{w(V(T))}{2}$. So, if r_l is not the required vertex, it it because one of its sub-trees has a weight $> \frac{w(V(T))}{2}$. Further the weight of all sub-trees of a leaf node is $0 < \frac{w(V(T))}{2}$. So, there must be a vertex v' on the path from the root to some leaf such for all connected components T_i of $T \setminus v'$ such that $w(V(T_I)) \leq \frac{w(V(T))}{2}$.

□

Definition 3 (W-balanced separator)**.** *Let G be a graph and $W \subseteq V(G)$. We call a set $X \subseteq G$ a $W-$balanced separator if for all connected components C of $G \setminus X$, $|W \cap C| \leq \frac{|W|}{2}$*

Lemma 4. *A tree has a $W-$balanced separator of size 1*

Proof. For any set W, define a function $w : V(G) \to \{0, 1\}$ as follows:

$$w(v) = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{if } v \notin W \end{cases}$$

Now, the proof of this lemma follows directly from the proof of *Lemma 2.* □

Using the notion of balanced separators, we are now ready to re-define $t-$decomposable graphs.

Definition 5 ($t-$decomposable)**.** *Let G be a graph. For all $H \subseteq V(G)$ and $W \subseteq H$, if there is a $W-$balanced separator in the graph induced on H of size $\leq t$, then G is said to be $t-$decomposable.*

Example: Let ξ_1 is the set of $1-$ decomposable graphs. The set of all forests $F = \xi_1$ (*Hint:* Check if a cycle is $1-$decomposable.)

Now, we are ready to define a decomposition of a graph which gives a tree-like structure which is desirable.

2.2 Decompositions into Trees

Definition 6 (Decomposition of a graph)**.** *The Decomposition of a graph G is a pair (T_r, f), where T is a tree rooted at r, $f : V(T_r) \to 2^{V(G)}$ is a function, and the following properties hold:*

1. $\bigcup_{w \in V(T_r)} f(w) = V(G)$.

2. *For every edge $(u, v) \in E(G)$, there is some $w \in V(T_r)$ such that $u, v \in f(w)$.*

3. *For a vertex $v \in G$, define $X_v := \{w | w \in V(T_r), v \in f(w)\}$. For all vertices $v \in G$, X_v forms a connected component in the tree T_r.*

Note that the above definition allows overlapping of vertices that are mapped to different nodes of the tree. Also, this type of decomposition also does not give an obvious measure of the graph like in the previous attempt when for every node w of the tree, $|f(w)| \leq t$. Hence, we will define a new measure for these kind of decompositions

Definition 7 (Width of a Decomposition)**.** *The width of a decomposition (T_r, f), denoted by $w(T_r, f)$, is defined as $w(T_r, f) = \max_{w \in V(T_r)}\{|f(w)| - 1\}$.*

Observation 8. *If a graph has at least* 1 *edge, the width of it's decomposition must be* > 0 *(property* 2 *of* Definition 6*).*

Example: Given a tree T, we can find a decomposition (T'_r, f) as follows:

- Let r_T be the root of T (if T is not rooted, select a vertex arbitrarily and assign it as the root).
- T' is a tree isomorphic to T under isomorphism g; $g(r) = r_T$.
- $f(r) = r_T$.
- $f(v) = \{g(v), g(u)\}$ where $g(u)$ is the parent node of $g(v)$.

It is easy to see that this is a decomposition of width 1.

While this type of decomposition seems very convenient, it is important to ask, whether for any $t-$decomposable graph, a decomposition of a bound width exists, and if it does, how we can efficiently obtain it. We shall answer the first question in the next section.

3 Existence of the Decompositions with Bounded Width

We shall prove, here, there is a decomposition for any $t-$decomposable graph of bounded width.

Theorem 9. *Every $t-$decomposable graph G has a decomposition $T = (T_r, f)$ of width $w(T) < 3t+4$*

If one tries proving it using induction on $|V(G)|$, one might proceed as follows:

Base Case: If $|V(G)| \leq 3t + 4$, define a decomposition (T, f) such that T has only one node w and $f(w) = V(G)$. Clearly, $w(T, f) < 3t + 4$.

Induction Step: Since G is $t-$decomposable, there exists a set balanced separator $S \subseteq G$ of size $\leq t$. Let $C_1, C_2, \ldots, C_l$ be the components of $G \setminus S$. Since all $C_1, C_2, \ldots, C_l \subsetneq G$, they must have decompositions of width $< 3t + 4$. But it is not clear how we extend them to find a decomposition for G without violating property *3* of *Definition 6*.

So, we shall, instead, prove the following stronger result:

Theorem 10. *Let W be any subset of a $t-$decomposable graph G such that $|W| \leq 2t+4$. There exists a decomposition $T = (T_r, f)$ of width $w(T_r, f) < 3t+4$ and $W \subseteq f(r)$.*

Proof. We shall prove this by induction on $|V(G)|$.

Base Case: For $|V(G)| \leq 3t+4$, construct T_r so that $V(T) = r$ and $f(r) = V(G)$. This is a trivial decomposition and $w(T) \leq 3t+4$.

Induction Step: Let G be a graph such that $V(G) > 3t+4$. Let $W \subseteq G$ such that $|W| \leq 2t+4$. Add vertices arbitrarily to W, if needed, so that $|W| = 2t+4$ (note that this does not alter implication of the claim in any way). Since G is $t-$decomposable, there is a $W-$balanced separator $S \subset G$ such that $|S| \leq t$. Let $C_1, C_2, \ldots, C_l$ be the connected components of $G \setminus S$.
Define $W_1, W_2, \ldots W_n$ as follows:

$$W_i = C_i \cap W \text{ for } 1 \leq i \leq l$$

Note that $|W_i| \leq \frac{|W|}{2}$ for $1 \leq i \leq l$ (follows from the definition of a $W-$balanced separator).
Define $G_1, G_2, \ldots, G_n$ as follows:

$$G_i = C_i \cup S \text{ for } 1 \leq i \leq l$$

For any i, $|W \cap G_i| \leq |W_i| + |W \cap S| \leq |W_i| + |S| \leq \frac{W}{2} + |S| = (\frac{2t+4}{2}) + (t) = 2t+2$. Now, since $|W| = 2t+4$, G_i doesn't contain at least one vertex of W, and hence, of G, i.e. $|V(G_i)| \lneq |V(G)|$. By induction hypothesis, the stated proposition holds for G_i, which is also $t-$decomposable. Take $U_i = W_i \cup S$. $|U_i| \leq |W_i| + |S| \leq 2t+2 \leq 2t+4$. So, there is a tree decomposition $T^i = (T^i_{r_i}, f_i)$ of G_i such that $U_i \subseteq f_i(r_i)$.
Now construct the decomposition (T, r) for G as follows:

$$V(T) = \bigcup_{i=1}^{l} V(T^i) \cup \{r\}$$

$$E(T) = \bigcup_{i=1}^{l} E(T^i) \cup \bigcup_{i=1}^{l} (r, r_i)$$

$$f(b) = \begin{cases} S \cup W & \text{if } b = r \\ f_i(b) & \text{if } b \in T_i \end{cases}$$

Now we show that T_r is a tree decomposition with the desired properties. Clearly, property 1 of tree decompositions hold for T_r. To see why property 2 holds, note the following:

- For every edge (u, v) such that u and v are both not in S, the property holds (induction, because u and v belong to the same connected component of $G \setminus S$) and for every edge (u, v) such that u and v are both in S, the property holds $(u, v \in f(r))$.
- For an edge (u, v), if $u \in S$ and $v \notin S$, Let C_i be the component of $G \setminus S$ which contains v. By induction hypothesis, there must exist a node n in the decomposition of $G \cup C_i$ such that $u, v \in f(n)$.

And for property 3:

- X_v forms a connected component if $v \notin S$ (induction).
- if $v \in S$, then $v \in f(r_i)$, since $S \subseteq U_i \subseteq f_i(r_i) = f(r_i)$. Since X_v forms a connected component in T_{r_i}(induction), it forms a connected component in T because $r_1, r_2, \ldots, r_l$ are connected to r.

Also, it is clear to check that $w(T) < 3t + 4$, since $w(T^i) < 3t + 4$ for $1 \leq i \leq l$ and $|f(r)| = |S \cup W| \leq |S| + |W| \leq (t) + (2t + 4) \leq 3t + 4$. Hence $w(T) < 3t + 4$

□

Now that we are sure of the existence of such decompositions, we will work towards computing them efficiently.