Advanced Graph Algorithms Jan-Apr 2014

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Lecturer: Saket Saurabh *Scribe: Diptapriyo Majumdar*

1 Overview

In the last lecture we have seen *tree decomposition* of a graph, *treewidth* of a graph. Then, we have learnt how to get an approximate tree decomposition from a t-decomposable graph.

In this lecture we have been reintroduced about these notions including one more terminology which is how to model a graph as a tree.

2 Main Section

Our main objective is to find an approximate tree decomposition of a t-decomposable graph.

2.1 Definitions

Definition 1. *A set $X \subset V[G]$ is called a $W-$**balanced separator of** G if for all connected componenet C of $G \setminus X$ we have that $|C \cap W| \leq \frac{|W|}{2}$*

Definition 2. *A graph $G = (V, E)$ is said to be $t-$decomposable if the following things happen: $\forall H \subset V[G]$ and $\forall W \subset H$, there is a $W-$balanced separator of size at most t in $G[H]$.*

Definition 3. ***Decomposition of a graph** $G = (V, E)$ is a pair (T, f) where $f : V(T) \to 2^{V[G]}$ that satisfies the following properties:*

(i) $\cup_{v \in V[T]} f(v) = V[G]$

(ii) $\forall e = \{u, v\} \in E[G]$, *there is a node* $w \in V[T]$ *such that* $\{u, v\} \subset f(w)$

(iii) $\forall v \in V[G]$, *let* $X_v = \{w | w \in V[T], v \in f(w)\}$. $T[X_v]$ *is connected.*

Note: $X_v = f^{-1}(v)$

Definition 4. ***A model of a graph** G **into** H is a function $\beta : V[G] \to 2^{V[H]} \setminus \{\emptyset\}$ such that the following properties are satisfied.*

(i) $H[\beta(v)]$ *is connected*

(ii) If $\{u, v\} \in E[G]$, *then* $\beta(u) \cap \beta(v) \neq \emptyset$

A model of a graph G into H has width $\omega(T,\beta) = max_{b\in V[H]}[|\{v \in V[G]|v \in \beta(b)\}| - 1]$

***Tree decomposition of a graph G** is a pair (T,β) where T is a tree and β is a model of G into T.*

Definition 5. ***Treewidth of a graph** G is denoted by $tw(G)$ and it is the minimum width $\omega(T,\beta)$ over all tree decomposition by $\omega(T,\beta)$.*

2.2 Properties of Tree Decompositions

Lemma 6. *If G is a tree, then $tw(G) \leq 1$*

Lemma 7. *Let G be a graph of $tw(G) = t$, then it is $(t+1)$-decomposable.*

Theorem 8. *If a graph G is t-decomposable, then $tw(G) \leq 3t + O(1)$.*

Theorem 9. *If a graph G has treewidth t, then it is $(t+1)$-decomposable.*

Proof. (of *Theorem 9*) Let G be a graph having with $tw(G) = t$.
Then, $\exists(T,\beta)$ with $t = \omega(T,\beta) = max_{b\in V[H]}[|\{v \in V[G]|v \in \beta(b)\}| - 1]$.
$\Rightarrow t + 1 \leq max_{b\in V[H]}[|\{v \in V[G]|v \in \beta(b)\}|$
Therefore G is $(t+1)$-decomposable. □

Theorem 10. *Let G be a t-decomposable graph and let $W \subset V(G)$ with $|W| \leq 2t+4$, then there exists a tree decompostition of width at most $3t+4$ where the root bag contains W.*

Proof. We prove it by induction on $V(G)$.
Base Case: If $|V(G)| \leq 3t+4$, then (T,β) is a tree decomposition where T contains a single bag which is the entire $V(G)$ itself.
Else We have to add necessary number of vertices so that we get a W' of size $2t+4$ exactly.
$|V(G)| \leq 2t+4$.
Since there exists a $W \subset V(G)$ such that $W \leq 2t+4$. Since G is t-decomposable, then there exists a W-balanced separator S with $|S| \leq t+1$.
Let $C_1, C_2, \ldots, C_l$ be the connected components of $G \setminus S$.
Define $W_i \forall i = 1,2,\ldots,l$ as $W_i = W \cap C_i$.
From the definition of W-balanced separator, we have that $|W_i| \leq \frac{|w|}{2}$
Define $G_i \forall i = 1,\ldots,l$ where $G_i = (G[C_i \cap S], W_i \cup S)$
$\forall i, |W \cap G_i| \leq |W_i| + |W \cap S \leq |W_i| + |S| \leq \frac{|W|}{2} + |S| = \frac{2t+4}{2} + (t+1) = 2t+3$

Now we have $|V_i| < |V|$. The separation property is inherited by the subgraph G_i and thus by induction hypothesis, G_i has a tree decomposition $(T_i, f_i) \forall i = 1,\ldots,l$ such that the followings are satisfied:

- $\omega(T_i, f_i) \leq 3t+3$
- $W_i \subset f(r_i)$

Now we construct a tree decomposition (T, f) by joining them together with a root r with $f(r) = W \cup S$.

$V(T) = \{r\} \cup \cup_{i \in \{1,2,\ldots,l\}} V(T_i)$
$E(T) = \cup_{i \in \{1,2,\ldots,l\}} E(T_i) \cup \{(r, r_i) | i \in \{1, 2, \ldots, l\}\}$
$f(t) = W \cup S$ if $t = r$
$= f_i(t)$ if $t \in T$ for some $i \in \{1, 2, \ldots, l\}$

The only vertices that any two of the graphs G_i have in common are contained in the separator S. Since $S \subset f(r_i)$ and $S \subset f(r)$, therefore condition (i) is satisfied. All edges of G that are not edges of any G_i have both endpoints in S and are therefore realized by $f(r)$. Thus condition (ii) is also satisfied. It is easy to see that condition (iiI) is also satisfied. Hence, (T, f) is a tree deomposition and it has width at most $3t + 4$ and its root bag contains W (Proved). □