Advanced Graph Algorithms Jan-Apr 2014

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1 Overview

In the last lecture, we proved the Schwarz-Zippel lemma and used it in intersting way by reducing the perfect matching problem to a Polynomial Identity Testing problem. Both, the Edmund's matrix for bipartite graphs, and the Tutte's matrix for general graphs can be used as a starting point and we have proved results for both of them.

In this lecture we, shall see a few ideas related to cycles in directed graphs. First, we want to compute whether or not a directed graph has a cycle. Though that can be easily done using BFS or DFS, we want to do something more interesting and reduce this problem to Polynomial Identity Testing. Then, we shall use similar ideas to compute the weight of the cycle in the graph that has the minimum weight among all cycles. We do not, however, show a way to compute the cycle itself.

2 Reducing Graph Problems to PIT

We begin by describing the following two problems

Problem 1:
Input: A directed graph G = (V,E).
Output: Does G have a directed cycle?

Problem 2:
Input: A directed graph G = (V,E) and a weight function $w : E \to \{0, 1, \ldots, W\}$.
Output: The weight of the cycle which has the least weight among all cycles in the graph.

We shall reduce both the above problems to that of Polynomial Identity Testing. First, we shall look at the first problem.

2.1 Reducing cycle detection to Polynomial Identiity Testing

For this problem we define a matrix M for a graph $G = (V, E)$ as follows:

$$M_{i,j} = \begin{cases} 1 & \text{if } (i = j) \\ x_{i,j} \times y^w & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Before we begin the proof, it helps to mention that every permutation σ can be thought of as a disjoint set of directed cycles in which each node corresponds to one element of the set, and for any 2 elements i, j, there is a directed edge from i to j iff $\sigma(i) = j$. Since, a permutation is a bijection from a set onto itself, every node has one edge into it and one egde out of it (these may be the same edge, in which case, an element is simply mapped to itself in σ). Hence, they form a disjoing set of cycles that we shall call *permutation cycles*

Lemma 1. *The graph has a directed cycle iff* $det(M) \not\equiv 1$

Proof. Since $det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$, here, we think of permutations in terms of *permutation cycles*. Let $C_1, C_2, \ldots, C_l$, be the cycles of a permutation σ. We say $(i, j) \in \sigma$ iff $\sigma(i) = j$ or there is an edge from i to j in some directed cycle of the permutation σ. Thus, we can write $det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{(i,j) \in \sigma} M_{i,j}$. Now, we observe that if the determinant contains any term other than 1(which corresponds to the identity permutation, where for every $i \leq n$, $\sigma(i) = i$), that it identically not equal to 0, that means there is at least one cycle of length ≥ 2.

Now we prove that every nontrivial (length ≥ 2) and non-zero (doesn't contain an edge (i, j) such that $M_{i,j} = 0$) cycle in a permutation σ corresponds to a cycle in the graph. This can easily be achieved since if $(i, j) \in \sigma$, then $\sigma(i) = j$, and since the term is nonzero, $M_{i,j} = x_{i,j}$ (it is not 1 since it is a non-trivial cycle), and hence, $(i, j) \in E(V)$. Thus every edge in the permutation corresponds to an edge in the graph.

The other direction is easy and follows from the previous observation. If there is a cycle in the graph, then there is a permutation $\sigma \in S$ that contains that cycle, which corresponds to the cycle on the same nodes in the graph, so at least one of its permutation cycles are non-trivial and non-zero, and that corresponds to one term in the determinant, that is not identicaly equal to 0 or 1. □

Now that we have the necessary and sifficient criterion for a cycle, we shall make use of Schwarz-Zippel lemma to check if the determinant is identically equal to one with a bound on the error depending on the size of the set we pick the values from.

2.2 Reducing Shortest Weight Cycle to Polynomial Identity Testing

Since the variables for every edge (i, j) are part of a product if they belong to the same permutation, which are, in essence, cycles from the graphs, it appears to be a good attempt at this problem, if we multiply each edge variable $x_{i,j}$ by y^w where y is a variable and w is the weight of edge (i, j).

So, we define the new matrix $M_{n \times n}$ for the given graph $G = (V, E)$ as follows:

$$M_{i,j} = \begin{cases} 1 & \text{if } (i = j) \\ x_{i,j} \times y^w & \text{if } (i,j) \text{ is an edge in } G \text{ of weight } w,\ i \neq j \\ 0 & \text{otherwise} \end{cases}$$

$M_{i,i} = 1$ for $1 \leq i \leq n$ and $M_{i,j} = x_{i,j} \times y^x$, $i \neq j$, if $(i,j) \in E(G)$ is an edge in the graph with weight w and 0 otherwise.

To simplify this problem further let us define a *Cycle Packing*

Definition 2. *Cycle Packing A* Cycle Packing *of a graph is a set of vertex disoint cycles such that every vertex belongs to some cycle.*

It is not always the case that such a packing is always exists for simple graphs. To take care of this, we think of vertices with a trivial edge to themselves are valid cycles. We can think of every vertex having an edge to itself of weight zero. Note that this does not violate the matrix M that we defined. We must not, however, immediately say that the weight of a minimum weight cycle is 0 and corresponds to a trivial cycle.

Another use of this definition is that every permutation corresponds to a unique vertex packing. We had showed, in the previous proof, that every non-trivial cycle in the permutation corresponds to a cycle in the graph. With this new additional feature, however, we can say that a trivial cycle in any permutation also corresponds to a cycle in a cycle packing.

The weight function that was used for edges, can be extended to a cycle packing. The weight of a cycle packing C, denoted by $w(C)$ is given by the sum of weights of all cycles in the cycle packing.

Observation 3. *Each summand in the summation of $det(M)$ is the weight of a cycle packing given by the corresponding permutation. Thus for every cycle C, there is a cycle cover whose weight is equal to $w(C)$.*

Now, we will find the minimum non-zero degree of y whose coefficient is not identicaly equal to zero. This will be equal to the degree of y in the sum $\sum_{\sigma \in S'_n} sgn(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$, where S'_n is the set of all permutations which correspond to some minimum weight packing. Before that, we prove that this is what we want.

Let $deg^*_x(P(x))$ is the minimum non-zero degree of x in a polynomial $A(x)$, i.e the minimum number p that is non-zero, such that x^p has a coefficient that is not identically equal to zero in the polynomial $A(x)$.

Define

$$H = deg^*_y(det(M[x] - 1))$$

.

Theorem 4. *Let D be a directed weighted graph with weights from the set $\{0, 1, 2, \ldots, W\}$. The weight of the cycle with minimum weight in a G is equal to H.*

Proof. let the weight of the cycle with minimum weight be w_{min}.
Suppose $H < w_{min}$. Then, from *Observation 3*, there is a Cycle Packing of weight H. Since this is not the packing in which all the vertices belong to trivial cycles, there must be a cycle in H, that is less than w_{min}. But w_{min} is, by definition, the weight of the cycle with the minimum weight. Hence $H \geq w_{min}$.

Suppose $H > w_{min}$. Take the following permutation: the permutation cycle corresponding to the cycle of weight w_{min}, and the rest of vertices (if any) in trivial cycles. This will give rise to a term where degree of y is w_{min}, which is $< H$ (assumption). But H, by definition is the least power of y in the polynomial $det(M[x] - 1)$. Hence $H \leq w_{min}$.

Hence $H = w_{min}$. □

What is remaining is how we calculate the determinant of the matrix M as a polynomial in y when all it's entries are monomials in y.

We cite here, the result from [1]

Theorem 5. *Let K be an arbitrary field and let $A = K[y]^{n \times n}$ be a matrix of degree d. Then $det(A)$ can be computed in $\mathcal{O}(n^{\omega}d)$ operations in K with a high probability.*

In the above proof, we have used only non-negative weights. However, the problem can be easily extended to a graph that may contain negative edges but no negative weight cycles. Let the weights of the graphs, be from a set $\{-W, -(W-1), \ldots, 0, 1, 2, \ldots, W\}$, we simply multiply each term in the matrix M by y^W. Now, since every vertex is included in the Cycle Packing, the least degree of y that has a coefficient that is not identically equal to zero will be $w_{min} + nW$, where w_{min} is the weight of the minimum weight cycle and n is equal to the number of edges. In this case, the minimum weight of the minimum-weight cycle can be obtained by subtracting nW from the obtained value.

References

[1] Marek Cygan, Harold N. Gabow, Piotr Sankowski: Algorithmic Applications of Baur-Strassen's Theorem: Shortest Cycles, Diameter and Matchings. FOCS 2012: 531-540