Advanced Graph Algorithms Jan-Apr 2014

Lecture 9 — January 22, 2014

Lecturer: Saket Saurabh *Scribe: Diptapriyo Majumdar*

1 Overview

In the last lecture we learnt to solve two problems by using Polynomial Identity Testing. One of them was to identify whether a given directed graph has a directed cycle or not. Other one was to find the weight of a directed cycle that has the smallest weight from a given weighted directed graph where weights of edges are assumed to be non-negative.

In this lecture we will see an important property of linear algebra that relates the determinant of a given skew symmetric matrix over reals with the *pfaffian* of that matrix.

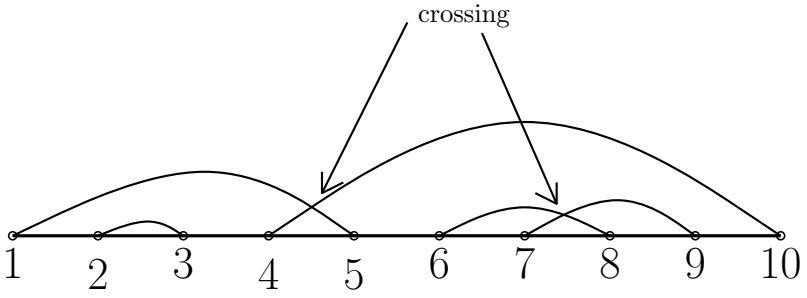
2 Main Section

2.1 Definitions

Definition 1. *Let $A_{n\times n}$ be a matrix over reals. A is said to be skew symmetric if and only if $A = -A^T$ where A^T is the transpose matrix of A.*

Definition 2. *Matching μ of $[n]$ is an unordered partition of n into $\frac{n}{2}$ unordered pairs. $\mu = (i_1, j_1), \ldots, (i_{\frac{n}{2}}, j_{\frac{n}{2}})$ where $i_k < j_k$ for all k.*

Definition 3. *We define $sign(\mu)$, we denote $\#\mu$ to be the number of crossings in the matching μ and $sign(\mu) = (-1)^{\#\mu}$. Crossing is demonstrated in the figure given below.*



2.1.1 Number of different Matchings

Let M be the set of all possible matchings. There are n elements. We fix one $k \in [n]$ for i_1, then we have $(n-1)$ choices to pick j_1. Then, we recursively call the same function for the rest of $(n-2)$ elements. Hence the recurrence is

$$T(n) = (n-1)T(n-2)$$

$$T(2) = 1$$

Solving the recurrence, we get $T(n) = (n-1)(n-3)\dots 1$ different matchings.

Definition 4. *Let A be a real skew-symmetric $n \times n$ matrix and n is even. Then, the $Pfaffian$ $Pf(A)$ is defined as*

$$Pf(A) = \sum_{\mu} sign(\mu) a_{\mu}$$

where we use the notation $a_\mu = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_{n/2} j_{n/2}}$.

We see that the terms in the right hand side are of the form $a_{i_1,j_1}.a_{i_2,j_2}\dots a_{i_{\frac{n}{2}},j_{\frac{n}{2}}}$ (with same or different signs). This creates a partition of n into $\frac{n}{2}$ pairs called a *matching* μ on $[n]$ writing $\mu = i_1 j_1, i_2 j_2, \dots, i_{\frac{n}{2}} j_{\frac{n}{2}}$ with $i_k < j_k$ for all k.

Now, we state the following theorem which we will prove in this lecture.

Theorem 5. *Given a real skew symmetric matrix $A_{n\times n}$ over reals where n is even. Then $det(A) = [Pf(A)]^2$ where $Pf(A)$ is called the Pfaffian of a matrix.*

2.2 Involution Property

We start with the usual expression for the determinant

$$det(A) = \sum_{\sigma \in S_n} sign(\sigma) a_\sigma \text{ where } a_\sigma = \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Let $B \subset S_n$ be those permutations that have at least one cycle of odd length. Define $\psi : B \to B$ as follows. For $\sigma \in B$, let $\sigma = \prod_{i=1}^{t} \sigma_i$ be the cycle decomposition and let σ_1 be a cycle of odd length that contains the smallest numbered vertex among the cycles of odd length and define

$$\psi(\prod_{i=1}^{t} \sigma_i) = \sigma_1^{-1} \prod_{i=2}^{t} \sigma_i$$

Then, ψ is an involution of B. If $\psi(\sigma) = \sigma$, then σ is the identity permutation. Since, $A_{k,k} = 0$ for all $k \in [n]$, therefore $a_{id} = 0$.

Now, we obtain

$$a_{\sigma_1} = \prod_{i \in \sigma_1} a_{i,\sigma_1(i)} = -\prod_{i \in \sigma_1} a_{\sigma_1(i),i} = -\prod_{i \in \sigma_1^{-1}} a_{i,\sigma_1^{-1}(i)} = -a_{\sigma_1^{-1}}$$

And we conclude that $a_\sigma = -a_{\sigma'}$ and thus $sign(\sigma)a_\sigma = -sign(\sigma')a_{\sigma'}$ since sign stays the same. Let $C = S_n \setminus B$. All ψ pairs cancel out. Therefore, we obtain

$$det(A) = \sum_{\sigma \in C} sign(\sigma) a_\sigma$$

Now, we define $\phi : M \times M \to C$ with $\phi|(\mu_1, \mu_2) \to \sigma$ whenever

$$sign(\mu_1) a_{\mu_1} . sign(\mu_2) a_{\mu_2} = sign(\sigma) a_\sigma$$

The pair (μ_1, μ_2) decomposes $[n]$ into vertex disjoint directed cycles. All cycles are of even length. Let σ be the permutation where $\sigma = \sigma_1 \sigma_2 \sigma_3 \dots \sigma_t$ be the cycle decomposition where where σ_i is a cycle of even length with $\sigma_i = (i_1, i_2, \dots, i_{2k})$ be such a cycle, then $(i_x, i_{x+1}) \in \mu_1$ if x is odd and $(i_x, i_{x+1}) \in \mu_2$ if x is even.

Lemma 6. *$\phi : M \times M \to C$ is a bijection.*

Proof. Consider any $\sigma \in C$. Now, all cycles of σ are of even length. Now, for any cycle $\sigma_i = (i_1, i_2, \dots, i_{2k})$ in the cycle decomposition of σ, then, $(i_x, i_{x+1}) \in \mu_1$ if x is odd and $(i_x, i_{x+1}) \in \mu_2$ if x is even. Therefore, we find (μ_1, μ_2) so that ϕ maps (μ_1, μ_2) to σ. Therefore, ϕ is a surjection. Also, if $(\mu_1, \mu_2) \neq (\mu_1', \mu_2')$, then without loss of generality lets assume that $\exists a_{i_1}$ such that $(a_{i_1}, a_{j_1}) \in \mu_1$ and $(a_{i_1}, a'_{j_1}) \in \mu_1'$ with $a_{j_1} \neq a'_{j_1}$. Then, consider $\sigma_1 = \phi(\mu_1, \mu_2)$ and $\sigma_2 = \phi(\mu_1', \mu_2')$. Then, $\exists$ an edge in σ_1 which is not in σ_2. Therefore, $\sigma_1 \neq \sigma_2$. Therefore, ϕ is an injection.
Therefore, ϕ is a bijection. □

Now, in order to prove the theorem, we need to show one to one correspondence between the terms of $[Pf(A)]^2$ and $det(A)$.

$$[Pf(A)]^2 = [\sum_{\mu_1} sign(\mu_1) a_{\mu_1}] . [\sum_{\mu_2} sign(\mu_2) a_{\mu_2}] = [\sum_{\mu_1, \mu_2} sign(\mu_1) . sign(\mu_2) . a_{\mu_1} . a_{\mu_2}]$$

$$det(A) = \sum_{\sigma \in C} sign(\sigma) a_\sigma$$

In order to show one to one correspondence between the terms of $[Pf(A)]^2$ and $det(A)$, we need to prove the following lemma.

Lemma 7. *$sign(\mu_1).a_{\mu_1}.sign(\mu_2).a_{\mu_2} = sign(\sigma).a_\sigma$ when $\phi(\mu_1, \mu_2) = \sigma$.*

Proof. ϕ maps (μ_1, μ_2) to $\sigma = \sigma_1 \sigma_2 \dots \sigma_t$ where $\forall j = 1, 2, \dots, t$ σ_j is a cycle of even length.

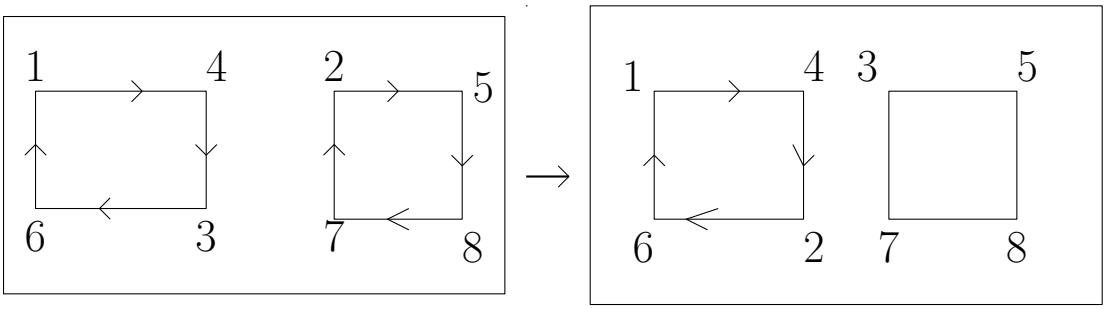
Consider a cycle $\sigma_i = (i_1, i_2, \dots, i_{2k})$ where $(i_x, i_{x+1}) \in \mu_1$ if x is odd, otherwise $(i_x, i_{x+1}) \in \mu_2$. Since ϕ maps (μ_1, μ_2) to σ, therefore, $a_{\mu_1} . a_{\mu_2} = (-1)^{e_\sigma} . a_\sigma$ where $e_\sigma = |\{i | \sigma(i) < i\}$.

Now, to prove the lemma, we have to prove $sign(\mu_1).sign(\mu_2).(-1)^{e_\sigma} = sign(\sigma)$ which is equivalent to proving $(-1)^{\#\mu_1 + \#\mu_2 + e_\sigma} = (-1)^t$ which is equivalent to proving $\#\mu_1 + \#\mu_2 + e_\sigma \equiv t (\text{mod } 2)$.

Now, in order to prove $\#\mu_1 + \#\mu_2 + e_\sigma \equiv t (\text{mod } 2)$ we need to prove the following lemma. □

Lemma 8. *Let $\sigma \in C$ and $\sigma = \phi(\mu_1, \mu_2)$ and $\hat{\sigma} \in C$ is the permutation that is obtained by interchanging the positions of i and $i+1$ in the cycle decomposition. Then, $\#\mu_1 + \#\mu_2 + e_\sigma \equiv \#\hat{\mu_1} + \#\hat{\mu_2} + e_{\hat{\sigma}} (mod\ 2)$ where $\phi(\hat{\mu_1}, \hat{\mu_2}) = \hat{\sigma}$.*

One step of the transformation is demonstrated in the figure. Here 2 has been exchanged with 3.



Proof. If i and $i+1$ are not next to each other in a cycle, then $e_\sigma = e_{\hat{\sigma}}$ remains unchanged after the transformation. Also, $\#\hat{\mu_1} = \#\mu_1 \pm 1$ and $\#\hat{\mu_2} = \#\mu_2 \pm 1$. Therefore, $\#\hat{\mu_1} + \#\hat{\mu_2} + e_{\hat{\sigma}} = \#\mu_1 \pm 1 + \#\mu_2 \pm 1 + e_\sigma = \#\mu_1 + \#\mu_2 + e_\sigma \pm 2$. Therefore, $\#\mu_1 + \#\mu_2 + e_\sigma \equiv \#\hat{\mu_1} + \#\hat{\mu_2} + e_{\hat{\sigma}} (\text{mod } 2)$.
If i is next to $i+1$ in a cycle. Then, either $\#\hat{\mu_1} = \hat{\mu_1} \pm 1$ or $\#\hat{\mu_2} = \hat{\mu_2} \pm 1$ but not both. And $e(\hat{\sigma}) = e(\sigma) \pm 1$.
Therefore, $\#\mu_1 + \#\mu_2 + e(\sigma) \pm 2 = \#\hat{\mu_1} + \#\hat{\mu_2} + e(\hat{\sigma})$. Therefore, $\#\mu_1 + \#\mu_2 + e_\sigma \equiv \#\hat{\mu_1} + \#\hat{\mu_2} + e_{\hat{\sigma}} (\text{mod } 2)$ (Proved).

□

If we apply such transformation finite number of times, then we get $\tilde{\sigma} = (1, 2, \ldots, k)(k{+}1, \ldots, h) \ldots (\ldots, n)$ and $\#\tilde{\mu_1} = \#\tilde{\mu_2} = 0$, $\tilde{\sigma} = e_{\tilde{\sigma}} = t$. Then, $\#\tilde{\mu_1} + \#\tilde{\mu_2} + e_{\tilde{\sigma}} \equiv t (\text{mod } 2)$ which means $\#\mu_1 + \#\mu_2 + e(\sigma) \equiv \#\tilde{\mu_1} + \#\tilde{\mu_2} + e_{\tilde{\sigma}}$ which we have proved in the last lemma. This finally proves $\#\mu_1 + \#\mu_2 + e_\sigma \equiv t (\text{mod } 2)$ and it finally establishes one to one correspondence between terms of $[Pf(A)]^2$ and $det(A)$ and finally proves the theorem.

References

[1] Martin Aigner: *A Course in Enumeration*

[2] Jeanette Nguyen: *Perfect Matching and Pfaffian Orientations, Bachelor Thesis*