Advanced Graph Algorithms Jan-Apr 2014

Lecture 10 — January 27, 2014

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1 Overview

In the last lecture we derived the following equality between the determinant and the Pfafian of any real skew-symmetric matrix $A = [a_{ij}]$.

$$det(A) = (Pf(A))^2$$

where, $Pf(A) = \sum_{\mu} sgn(\mu)a_{\mu}$ and μ is a partition of $[n]$ into $\frac{n}{2}$ pairs such that $\mu = (i_1, j_1), \ldots, (i_{\frac{n}{2}}, j_{\frac{n}{2}})$ and $i_k < j_k$ for all $1 \leq k \leq \frac{n}{2}$; and $a_{\mu} = a_{i_1 j_1} \ldots a_{i_{\frac{n}{2}} j_{\frac{n}{2}}}$. The aim of this lecture is to exploit the formula for Pfafian to find the number of perfect matchings in planar graphs. Thus, the task of counting the number of perfect matchings in planar graphs is reduced to calculating the determinant of the corresponding matrix of the graph.

2 How does Pfafian help in calculating perfect matchings in graphs

Let $M = [m_{ij}]$ be the Tutte's matix of a graph G. Then Pfafian of M,

$$Pf(M) = \sum_{\mu} sgn(\mu)a_{\mu} \tag{1}$$

Each non-zero summand in (1) represents one perfect matching in the graph. Thus, the number of non-zero summands in (1) gives us the number of perfext matchings in G. To find the number of non-zero summands it is enough to ensure that either each summand in (1) is positive (i.e. 1) or each summand is negative (i.e. -1). Then $|Pf(M)|$ gives the number of perfect matchings in G.

To achieve this goal let us arbitarily choose an orientation (direction) for each edge in G and construct a new adjacency matrix for G, say $A = [a_{ij}]$, as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is the oriented edge} \\ -1 & \text{if } (j,i) \text{ is the oriented edge} \\ 0 & \text{otherwise} \end{cases}$$

For any graph G, there are $2^{|E(G)|}$ such orientations. An orientation is called **Pfafian orientation** if $sgn(\mu)a_{\mu} = 1$ for all μ or $sgn(\mu)a_{\mu} = -1$ for all μ.

Thus, if one could get a Pfafian orientation for a graph then the task of counting perfect matchings in the graph is reduced to calculating the Pfafian and hence to finding the determinant of the oriented graph's adjacency matrix. The number of perfect matchings is then $\sqrt{det(A)}$.

Since counting number of perfect matchings in general graphs is known to be $\#P$-Hard [], one may not hope to efficiently count the number of perfect matchings in general graphs. But it turns out

that one can find a Pfafian orientation for planar graphs (refer Section 4). Thus one can count number of perfect matchings in planar graphs.

3 Prelims to Planar Graphs

Definition 1 (Planar Graph)**.** *A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a* **plane graph** *or planar embedding of the graph.*

Definition 2 (Euler's Formula)**.** *Euler's formula states that if G is a finite, connected, planar graph and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region), then $v - e + f = 2$.*

Definition 3 (Bridge)**.** *A bridge is an edge whose deletion makes the graph disconnected.*

Definition 4 (Clokwise-oriented edge with respect to a face)**.** *Consider a face f. Denote by B_f the set of boundary (directed) edges of f. We say that $e \in B_f$ is clockwise-oriented with respect to the face f if f lies on the right of e as we move along e in the direction of e.*

4 Counting number of perfect matchings in planar graphs

Let ϕ be a function such that $\phi : M \times M \to C$, where $M = \{\mu | \mu$ is a matching of $[n]\}$ and $C = \{\sigma \in S_n | \sigma = \sigma_1 \dots \sigma_t$ and each σ_i is of even length $\}$. Define $\phi(\mu_1, \mu_2) = \sigma$ whenever $sgn(\mu_1)a_{\mu_1} sgn(\mu_2)a_{\mu_2} = sgn(\sigma)a_\sigma$. From Lemma 6 of Lecture 09, ϕ is a bijection.

Recall that to get a Pfafian orientation we need to ensure that either $sgn(\mu)a_\mu = 1$ for all μ or $sgn(\mu)a_\mu = -1$ for all μ. To ensure this it is enough to ensure that $sgn(\sigma)a_\sigma = 1$ or 0 for all σ(From the definition of ϕ, $sgn(\sigma)a_\sigma = 1$ if and only if either $sgn(\mu_1)a_{\mu_1}$ and $sgn(\mu_2)a_{\mu_2}$ are both 1 or $sgn(\mu_1)a_{\mu_1}$ and $sgn(\mu_2)a_{\mu_2}$ are both -1).

Lemma 5. *Let $G = (V, E, F)$ be a simple connected plane graph. Then there exists an orientation such that for any face (except possibly the outer face), the number of clockwise oriented edges is odd.*

Proof. We prove this using induction on the number of faces ($|F|$).

Base Case ($|F| = 1$) **:** In this case, arbitarily orient one of the edges in clockwise direction with respect to the inner face and orient others in anti-clockwise direction with respect to the inner face.

Induction Step : Let us assume that lemma holds for any graph with $|F| = k$. Let G be a graph with $|F| = k+1$. Then there exists an edge e on the outer face such that $G \setminus e$ is not disconnected. It is because if every edge on the outer face disconnects the graph then each edge is a bridge and hence the graph is a tree for which the lemma trivially holds. Since each non-bridge edge is shared by 2 faces, the number of faces in $G \setminus e$ is k. By the induction hypothesis $G \setminus e$ has an orientation where every face (except possibly the outer face) has odd number of clockwise oriented edges. The required orientation for G is as follows: orient all edges of G except e as oriented in $G \setminus e$ and orient e clockwise with respect to the inner face containing e if number of clockwise oriented edges (except e) on this face is even and anticlockwise otherwise. □

Lemma 6. *Let $G = (V, E, F)$ be a simple planar graph without bridges, oriented according to Lemma 5. Let C be a cycle. Then the number of edges oriented clockwise in C has opposite parity to the number of vertices in the interior of C.*

Proof. Let $H = (V^{'}, E^{'}, F^{'})$ be the graph induced on the vertices of C and vertices in the interior of C. Let the length of the cycle C be x, the number of vertices in the interior of C be p, the number of edges in H excluding the edges of C be $e^{'}$ and the number of internal faces in H be f. Since G is planar and H is a subgraph of G, H is planar and hence, Euler's formula holds for H.

$$|V^{'}| - |E^{'}| + |F^{'}| = 2$$

$$p + x - e^{'} - x + f + 1 = 2$$

$$p - e^{'} + f = 1 \tag{2}$$

Let c_0 be the number of edges that are clockwise oriented in C. For each internal face B_i of H, let c_i be the number of edges that are clockwise oriented in B_i. Note that each c_i is odd because of Lemma 5.

Since every internal edge in H is shared by 2 internal faces, each internal edge is clockwise oriented for exactly one face. Therefore we have,

$$\sum_{i=1}^{f} c_i = e^{'} + c_0 \mod 2$$

Since each c_i is odd, it follows that

$$f = e^{'} + c_0 \mod 2$$

From 2, we have

$$1 + e^{'} - p = e^{'} + c_0 \mod 2$$

Hence,

$$1 + p = c_0 \mod 2$$

Therefore, the number of edges oriented clockwise in C has opposite parity to the number of vertices in the interior of C.

□

Theorem 7. *Let $G = (V, E, F)$ be a simple plane graph without bridges. Then the orientation given in Lemma 5 is Pfafian.*

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of the oriented graph. To show that the orientation given by Lemma 5 is Pfafian, it is enough to show that $sgn(\sigma)a_\sigma = 1$ for all $\sigma \in C$, where $C = \{\sigma | \sigma = \sigma_1 \dots \sigma_t$ and all σ_i 's are even cycles $\}$.

Let $\sigma = \sigma_1 \dots \sigma_t$ be the factor of σ and $a_\sigma = a_{\sigma_1} \dots a_{\sigma_t}$. Hence we need to show that $(-1)^t a_{\sigma_1} \dots a_{\sigma_t} = 1$ i.e. $a_{\sigma_i} = -1$ for all $1 \leq i \leq t$.

Since a_σ partitions the vertices of the graph into disjoint even cycles $(a_{\sigma_1}, \dots, a_{\sigma_t})$ and the graph is plane, the number of vertices in the interior of the cycle a_{σ_i} in the planar embedding of G is even Hence by Lemma 6 the number of clockwise oriented edges in a_{σ_i} are odd. Since a_{σ_i} is an even cycle, the number of anti-clockwise oriented edges is also odd. Hece,$a_{\sigma_i} = -1$. □