**Advanced Graph Algorithms** Jan-Apr 2014

Lecture 11 — January 28, 2014

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# 1 Overview

In this lecture we will revisit the problem of finding perfect matching in non-bipartite graphs. We will study the algebraic approach to solve this problem. In the earlier lectures, we have seen the algorithm to decide whether a graph has perfect matching or not. In this, we will present $O(n^{w+2})$ and $O(n^{w+1})$ order algorithm to find the perfect matching. Before attacking perfect matching problem, we will state few well known results from Linear Algebra.

# 2 Ideas from Linear Algebra

**Theorem 1.** *If $M$ is real skew symmetric matrix, then*

$$det(M) = [Pf(M)]^2 \tag{1}$$

*where determinant and pfaffian of matrix $A$ is given as follows*

$$det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$$

$$pf(M) = \sum_{\mu} sgn(\mu) m_{\mu}$$

*where $S_n$ is permutation group, $\mu$ is matching on $\{1, 2, \ldots, n\}$ and $sgn(\mu) = (-1)^{\#crossing}$.*

**Definition 2** (Rank)**.** *For any given matrix $M$, rank of $M$ is defined as dimension of largest square sub-matrix $M_{sub}$, such that $det(M_{sub}) \neq 0$.*

**Notation** Let $M$ be an $n \times n$ matrix and $X, Y$ be subsets of $\{1, 2, \ldots, n\}$. $M[X, Y]$ or $M_{XY}$ will denote the submatrix of $M$ obtained by choosing the rows of $M$ corresponding to indices in $X$ and columns corresponding to indices in $Y$. Similarly $M^{XY}$ will donate the submatrix of $M$ obtained by deleting rows of $M$ corresponding to indices in $X$ and columns corresponding to indices in $Y$.

**Theorem 3** (Jacobi's Theorem)**.** *Let $M$ be non singular matrix. For any $I, J \subset V$, $|I| = |J|$*

$$det(M[I, J]) = det(M) \times det(M^{-1}[V - J, V - I])$$

*Proof.* For given matrix $M$, let $M^*$ be its inverse. Write $M$ and $M^*$ in the following way

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad M^* = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}$$

where $D = M[I, J]$ and $D^* = M^*[I, J]$. Now consider the following product.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} = \begin{bmatrix} AA^* + BC^* & AB^* + BD^* \\ CA^* + DC^* & CB^* + DD^* \end{bmatrix} = \begin{bmatrix} I_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & I_{q\times q} \end{bmatrix}$$

using this relations, we simplify

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} A^* & 0_{p\times q} \\ C^* & I_{q\times q} \end{bmatrix} = \begin{bmatrix} AA^* + BC^* & B \\ CA^* + DC^* & D \end{bmatrix} = \begin{bmatrix} I_{p\times p} & B \\ 0_{q\times p} & D \end{bmatrix}$$

Hence

$$\begin{aligned} det(M) * det(A^*) &= det(D) \\ \Rightarrow det(M) * det(M^{-1}[I', J']) &= det(M[I, J]) \end{aligned}$$

□

The following theorem is due to Forbenius we state it here without proof.

**Theorem 4** (Forbenius)**.** *Let $M$ be an $n \times n$ skew-symmetric matrix, and let $X$ and $Y$ be subsets of $\{1, 2, \ldots, n\}$ such that $|X| = |Y| = rank(M)$. Then,*

$$det(M_{XX}) \times det(M_{YY}) = (-1)^{|X|} det(M_{XY})$$

**Lemma 5.** *Let $M$ be an $n \times n$ skew-symmetric matrix, and let $X$ and $Y$ be subsets of $\{1, 2, \ldots, n\}$ such that $|X| = |Y| = rank(M)$. If $rank(M[X, Y]) = r$ then $rank(M[X, X]) = r$.*

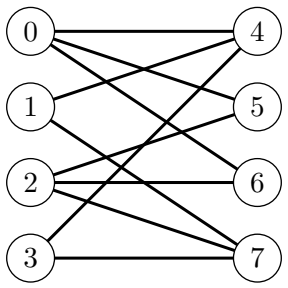
## 3 Edmonds and Tutte Matrix

**Definition 6** (Edmonds Matrix [MR95])**.** *For given balanced bipartite graph $G(U, V, E)$ with sets of vertices $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ Edmonds matrix $A$ of is defined by*

$$A_{ij} = \begin{cases} x_{ij} & (u_i, v_j) \in E \\ 0 & (u_i, v_j) \notin E \end{cases}$$

*where the $x_{ij}$ are indeterminates.*

Consider the following example



The Edmond matrix corresponding to above graph is

$$A = \begin{pmatrix} x_{04} & x_{05} & x_{06} & 0 \\ x_{14} & 0 & 0 & x_{17} \\ 0 & x_{25} & x_{26} & x_{27} \\ x_{34} & 0 & 0 & x_{37} \end{pmatrix}$$

The determinant of $A$ is given as

$$det(A) = -x_{06}x_{17}x_{25}x_{34} + x_{05}x_{17}x_{26}x_{34} + x_{06}x_{14}x_{25}x_{37} - x_{05}x_{14}x_{26}x_{37}$$

Note that every term in $det(A)$ corresponds to some element in $S_n$, which is perfect matching. For example the first term corresponds to the element $(0,6)(1,7)(2,5)(3,4)$ in $S_n$.

**Lemma 7.** *For given balanced bipartite graph $G$ and corresponding Edmond's matrix $A$, $G$ has perfect matching iff $det(A) \not\equiv 0$.*

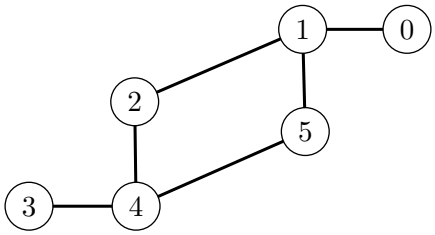
We generalize this concept of Edwards matrix for non-bipartite graphs by Tutte matrix.

**Definition 8** (Tutte Matrix)**.** *For any undirected graph $G(V, E)$, such that $|V| = n$. Tutte matrix $T$ of a graph $G$ is an $n \times n$ matrix with entries*

$$T_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in E \text{ and } i < j \\ -x_{ji} & \text{if } (i,j) \in E \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

*where the $x_{ij}$ are indeterminates.*

Consider the following example




The corresponding Tutte matrix

$$T = \begin{pmatrix} 0 & x_{01} & 0 & 0 & 0 & 0 \\ -x_{01} & 0 & x_{12} & 0 & 0 & x_{15} \\ 0 & -x_{12} & 0 & 0 & x_{24} & 0 \\ 0 & 0 & 0 & 0 & x_{34} & 0 \\ 0 & 0 & -x_{24} & -x_{34} & 0 & x_{45} \\ 0 & -x_{15} & 0 & 0 & -x_{45} & 0 \end{pmatrix}$$

In this case, determinant of matrix $T$ is zero and it is easy to check that there is no perfect matching in $G$. We stated and proved the following theorem in earlier classes.

**Lemma 9.** *For given graph $G$, let $T$ be corresponding Tutte matrix. $G$ has perfect matching iff $det(T) \not\equiv 0$.*

## 4 Relation between Rank and Perfect Matching

In the previous section, we saw that determinant of matrix Tutte matrix can we used to check the existence of perfect matching. In this section, we analysis it farther to get more information about matching.

The following generalization of Tutte's theorem was first given by Lovasz [LP86] and later the simplified version was presented in this paper [RV89] of by Rabin and Vazirani.

**Theorem 10** (Lovasz)**.** *Given graph $G$ and corresponding Tutte matrix $T$, $G$ has maximum matching of size $m$ iff $rank(T) = 2m$.*

*Proof.* We prove the theorem in two parts

*(i) To prove $rank(T) \geq 2m$.* Choose any matching of size $m$, and let $U \subset V$ be the set of vertices matched by it. Let $G'$ be the subgraph of $G$ induced on $U$, $G'$ has a perfect matching. Since $T_{UU}$ is the Tutte matrix of $G'$, $T_{UU} \not\equiv 0$. Since $|U| = 2m, Rank(A) \geq 2m$.

*(ii) To prove $rank(A) < 2m$.* Suppose $rank(T) = k$. Let $T_{XY}$ be a non-singular submatrix of $T$, with $|X| = |Y| = k$. We want to show that $G$ has a matching of size at least $k/2$. Notice that if $A_{XY}$ is symmetrically located, i.e. $X = Y$, we are done. (since the restriction of G to the vertices in $X$ must have a perfect matching by Tutte's theorem). $T_{XY} \not\equiv 0$ and $|X| = |Y| = rank(T)$, by Frobenious Theorem, $T_{XX} \not\equiv 0$. Since this is a symmetrically located submatrix of $T$, by previous argument, $G$ has matching of size $k/2$.

Hence $rank(T) = 2\times$ (size of maximum matching in graph $G$)

□

In the above example, rank of $T$ is 4 and the maximum matching of size 2 is present in graph $G$.

**Lemma 11.** *Let $T$ be $n \times n$ Tutte matrix of graph $G$. Let $\tilde{T}$ be an evaluation of $T$, with entries chosen uniformly at random from $\{1, 2, \ldots, n\}$ then $Rank(T) = Rank(\tilde{T})$ with probability $\geq 1/2$.*

*Proof.* Let $X \subset \{1, 2, \ldots, n\}$ such that $det(T[X, X]) \not\equiv 0$. $T[X, X]$ is skew symmetric matrix and we know that $det(T[X, X]) = Pf(T[X, X])^2$ this means, $det(T) \not\equiv 0 \Leftrightarrow Pf(T) \not\equiv 0$. But by the definition of Pfaffian, $Pf(T[X, X])$ is polynomial of degree $n/2$.

Now we substitute values for indeterminates chosen uniformly at random from $\{1, 2, \ldots, n\}$ then by Schwartz-Zippel lemma,

$$Pr(\tilde{T}[X, X] \neq 0) \geq 1/2$$

□

## 5 Algorithm to find Perfect Matching

Now we are in position to present the algorithm to find the perfect matching in the given non-bipartite graph. For given graph we construct corresponding Tutte matrix $T$. Let $\tilde{T}$ be an evaluation

of $T$, with entries chosen uniformly at random from $S$. We will define $S$ depending on the required upper bound on probability of error.

**Input:**$G(V, E)$
**Output:**Perfect matching in $G$, if exists
T $\leftarrow$ Tutte Matrix corresponding to G
**if** $det(T) \equiv 0$ **then**
  **return** *No Perfect Matching*
**end**
**for** *every edge* $(u, v) \in E$ **do**
  T[u, v] $\leftarrow 0$
  T[v, u] $\leftarrow 0$
  **if** $det(\tilde{T}) == 0$ **then**
    T[u, v] $\leftarrow x_{uv}$
    T[v, u] $\leftarrow -x_{uv}$
  **else**
    E $\leftarrow E \backslash \{(u, v)\}$
  **end**
**end**
**return** $E$

**Algorithm 1:** Finding Perfect Matching in non-bipartite graph

By the previous lemma $Pr(Error) \leq 1/2$ at every step. Since the $for$ loop will run for at most $n^2$ times, $Pr(Error) \leq n^2 * (n/|S|) = n^3/|S|$. Hence we select $S = \{1, 2, \ldots, 2 * n^3\}$ to bound $Pr(Error)$ by 1/2.

Since we are doing it for every possible edge, this algorithm will work in $O(n^{w+2})$. We can improve this algorithm further to achieve the upper bound of $O(n^{w+1})$. In the above algorithm we are spending $O(n^2)$ time to find edges which will not be part of Perfect Matching. To improve running time, we will spend $O(n)$ time to find edges which will be part of perfect matching.

**Lemma 12.** *For given bipartite graph $G$ and corresponding Edmond matrix $A$, an edge $(u_j, v_i) \in E(G)$, is in perfect matching iff $A^{-1}[i, j] \neq 0$.*

*Proof.* For skew symmetric, non singular metric $A$, adjugate of $A$ is defined as

$$adj(A)_{ij} = (-1)^{i+j} det(A^{ji}) \tag{2}$$

and by Cramer's rule

$$A^{-1}_{ij} = \frac{adj(A)_{ij}}{det(A)} \tag{3}$$

Hence if $A^{-1}_{ij} = 0 \Leftrightarrow adj(A)_{ij} = 0$, which implies that if we delete this edge, there will not be perfect matching in remaining graph. Hence proved. □

With the similar argument we can prove the following lemma.

**Lemma 13.** *For given graph $G$ and corresponding Tutte matrix $T$, an edge $(u_i, u_i) \in E(G)$, is in perfect matching iff $\tilde{T}^{-1}[i, j] \neq 0$.*

We this lemma, we now present the algorithm to find maximum perfect matching in non-bipartite graph in $O(n^{w+1})$.

**Input:**$G(V, E)$
**Output:**Perfect matching in $G$ if exist otherwise message – No Perfect Matching
T $\leftarrow$ Tutte Matrix for given graph
**if** $det(T) \equiv 0$ **then**
  **return** *No Perfect Matching*
**end**
PM $\leftarrow \phi$  //To store edges in perfect matching
**while** $|PM| \neq n$ **do**
  (i, j) $\leftarrow$ Find first non zero entry in $\tilde{T}^{-1}$
  Construct $\tilde{T}^{ij}$
  **if** $det(\tilde{T}^{ij}) == 0$ **then**
    PM $\leftarrow PC \cup \{u_i, u_j\}$
  **end**
**end**
**return** $PM$

**Algorithm 2:** Finding Perfect Matching in non-bipartite graph

## References


[LP86] L. Lovász and M.D. Plummer. *Matching Theory.* Akadémiai Kiadó, Budapest, 1986. Also published as Vol. 121 of the North-Holland Mathematics Studies, North-Holland Publishing, Amsterdam.

[MR95] Rajeev Motwani and Prabhakar Raghavan. *Randomized Algorithms.* Cambridge University Press, New York, NY, USA, 1995.

[RV89] M. O. Rabin and V. V. Vazirani. Maximum matchings in general graphs through randomization. *J. Algorithms*, 10(4):557–567, December 1989.