Advanced Graph Algorithms Jan-Apr 2014

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1 Overview

In the last lecture we learn about matroids and saw some examples of it. Continuing with an example we move to solve some interesting problems of graph theory using matroid intersection and matroid parity problem.

2 Problems on Matroid and Using Them

2.1 Transversal Matroids

Recall from the last lecture, transversal matroid $M = (E, \mathcal{I})$ of G has $E = A$ as its ground set, where G is a bipartite graph with the vertex set $V(G)$ being partitioned as A and B and

$$\mathcal{I} = \{X \mid X \subseteq A, \text{there is a matching that covers } X\}$$

The following shows the incidence matrix of a bipartite graph.

$$T = \begin{array}{l} \\ \textit{rows are} \\ \textit{indexed by} \\ \textit{vertices} \\ \textit{of} \\ B \end{array} \begin{array}{c} \begin{array}{ccccc} \textit{columns are} & \textit{indexed} & \textit{by} & \textit{vertices} & \textit{of}\, A \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{\ell 1} & x_{\ell 2} & x_{\ell 3} & \cdots & x_{\ell m} \end{pmatrix}_{\ell \times m} \end{array} \quad |A| = m \text{ and } |B| = \ell$$

$$t_{i,j} = \begin{cases} x_{i,j} & if (a_i, b_j) \in E(G) \\ 0 & otherwise \end{cases}$$

$x_{i,j}$ are indeterminants. All of them are independent variables. Next we show that this matrix indeed gives the represention for transversal matroids.

2.1.1 Transversal Matroid is Representable

Transversal Matroid is Representable over $\mathbb{F}[\vec{z}]$

Proof. Forward direction:
Suppose some subset $X = \{a_1, ..., a_q\}$ is independent. Then there is a matching that saturates X. Let $Y = \{b_1, b_2, ..., b_q\}$ be the endpoints of this matching and $a_i b_i$ are the matching edges. Consider $T[Y, X]$ a submatrix with rows in Y and columns in X. Consider the determinant of T [Y, X] then we have a term $\prod_{i=1}^{q} z_{ii}$ which can not be cancelled by any other term. So these columns are linearly independent.
Reverse direction:
Suppose some subset $X = \{a_1, \cdots, a_q\}$ of columns is independent in T. Then there is a submatrix of $T[*, X]$ that is nonsingular, say $T[Y, X]$. Consider the determinant of $T[Y, X]$

$$det(T[Y, X]) = \sum_{\pi \in S(Y)} sgn(\pi) \prod_{i=1}^{q} z_{i\pi(i)} \neq 0$$

This implies that we have a term $\prod_{i=1}^{q} z_{i\pi(i)} \neq 0$ and this gives us that there is a matching that saturates X in and hence X is independent.

In this direction we remove the indeterminants x_{ij}s. □

2.1.2 Finding the Representation

Removing x_{ij}
To remove the x_{ij} we assign x_{ij} from values in finite field F of size P uniformly at random. What should be the upper bound on P ? What is the probability that the randomly obtained T is a representation matrix for the transversal matroid. To answer this we use Schwartz-Zippel Lemma.

Schwartz-Zippel Lemma

Theorem 1. *Let $p(x_1, x_2, \cdots, x_n)$ be a non-zero polynomial of degree d over some field $\mathbb{F}$ and let S be an N element subset of $\mathbb{F}$. If each x_i is independently assigned a value from S with uniform probability, then $p(x_1, x_2, \cdots, x_n) = 0$ with probability at most $\frac{d}{N}$*

We think $det(T[Y, X])$ as polynomial in x_{ij}'s of degree at most $n = |A|$.
Probability ($det(T[Y, X]) = 0) < \frac{n}{P}$.
There are 2^n independent sets in A and thus by union bound, probability that not all of them are independent in the matroid represented by T is at most $\frac{2^n n}{P}$. Thus probability that T is the representation is at least $1 - \frac{2^n n}{P}$.
Take P to be some field with at least $2^n n 2^n$ elements. Size of this representation with be $n^{O(1)}$ bits.

This way we can find the representation in polynomial time using randomisation.

Open Problem Can we find the representation of transversal matroids in polynomial time using deterministic algorithm?

2.2 Two Problem Definitions

n-Matroid Intersection Given matroids $M_1 = (E, \mathcal{I}_1); M_2 = (E, \mathcal{I}_2); \cdots; M_n = (E, \mathcal{I}_n)$ on the same ground set E, and a number k, does there exist a set $X \subseteq E$ such that $|X| > k$ and $X \in \mathcal{I}_i$ for $1 \leq i \leq n$?

Matroid Pairity Given a matroid $M = (E, \mathcal{I})$, whose elements are partitioned into m pairs. (hence, $|E|$ is even).
Objective: Find the maximum cardinality collection of pairs, $X \subseteq E$ such that $X \in \mathcal{I}$?

Matroid Parity (Linear)
Given a matrix whose elements are partitioned into m pairs.

$$A = \begin{pmatrix} \overbrace{a_{11}\ b_{11}} & \overbrace{a_{12}\ b_{12}} & \cdots & \overbrace{a_{1m}\ b_{1m}} \\ a_{21}\ b_{21} & a_{22}\ b_{22} & \cdots & a_{2m}\ b_{2m} \\ a_{31}\ b_{31} & a_{32}\ b_{32} & \cdots & a_{3m}\ b_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}\ b_{r1} & a_{r2}\ b_{r2} & \cdots & a_{rm}\ b_{rm} \end{pmatrix}_{r \times 2m} \qquad r = \text{ rank of the matroid}$$

Objective: Find the maximum cardinality collection of pairs, such that unoin of columns of these pairs are linearly independent. ?

2.3 Problems

Question:1 *Phrase bipartite matching problem as intersection of two matroids*

Let $M_1 = (E, \mathcal{I}_1)$ where
$I_1 = \{X \subset E \mid \text{if } (i_1, j_1) \in X \text{ and } (i_2, j_2) \in X \text{ and } i_1, i_2 \in A \text{ and } j_1, j_2 \in B \text{ and } i_1 \neq i_2\}$
that is $\forall x_i \in X$ every x_i are incident on different vertex on A and
$M_2 =$ (E, $\mathcal{I}_2$) where
$I_2 = \{X \subset E \mid \text{if } (i_1, j_1) \in X \text{ and } (i_2, j_2) \in X \text{ and } i_1, i_2 \in B \text{ and } j_1, j_2 \in A \text{ and } i_1 \neq i_2\}$

We want to show that set family M_1 and M_2 are matroids.

Proof. M_1 satisfies (I2), since if all the edges in X are incident on different vertices of A then all its subsets are also incident on different vertex of A.
Suppose (I3) does not hold. That is there is set X_1 and $X_2 \in \mathcal{I}$. $|X_1| < |X_2|$. For all edges e $\in X_2 \setminus X_1$. e is incident on a vertex v such that $\exists$ an edge $e' \in X_1 \setminus X_2$. e' is also incident on v. But since $|X_2| > |X_1|$ and all edges in X_2 are incident on differet vertices then $\exists$ atleast one vertex u such that there is no edge $(u, v) \in X_1$. Then we can extend X_1 using that edge, Hence (I3) holds.

Therefore M_1 is a matroid. Using the same reason M_2 is also a matroid. □

Therefore $M_1 \cap M_2$ gives the matching in the bipartite graph.

Question:2 *Phrase matchng in general graph as matroid parity problem*

We represent the graph as the following matrix.

$$A = \begin{array}{c} \\ 1 \\ \vdots \\ u \\ v \\ \vdots \\ |V| \end{array} \begin{array}{c} \overbrace{e_{1_1}\ e_{1_2}} \quad \overbrace{e_{2_1}\ e_{2_2}} \quad \cdots \quad \overbrace{e_{|E|_1}\ e_{|E|_2}} \\ \begin{pmatrix} 0\,0 & & \cdots & \\ \vdots & \vdots & \vdots & \vdots \\ 1\,0 & & \cdots & \\ 0\,1 & & \cdots & \\ \vdots & \vdots & \vdots & \\ & & \cdots & \end{pmatrix}_{|V|\times|2E|} \end{array} \qquad a_{i,j} = \begin{cases} 1 & \text{if } i = u, j = e_{l_1}(u,v) = e_l \in E \\ 1 & \text{if } i = v, j = e_{l_2}, (u,v) = e_l \in E \\ 0 & \text{otherwise} \end{cases}$$

that is we take a $|V| \times |2E|$ matrix, we take two columns for each edge. This partitions the columns of the matrix in E pairs.
If two coulms are linearly dependent then they are same. So the i^{th} row have only one 1 if e_{k_1} and e_{k_2} are amongst the linearly independent columns and $(i,j) = e_k \in E$. Hence every vertex in the subgraph formed from these pair of linearly independent columns has degree atmost one, so it corresponds to a matching. Hence the union of columns of these pairs which are linearly independent gives the matching.

Question:3 *How do you test if a graph has two edge disjoint spanning trees*

Let G(V,E) be the graph. $M_1 = (E, \mathcal{I}_1)$ be the graphic matroid of G and $M_2 = (E, \mathcal{I}_2)$ be the co-graphic matroid of G.

If M_1 and M_2 has an intersection of size $|V| - 1$. Call this set of edges S. Then $S \in \mathcal{I}_1$ i.e., S forms spanning trees of G. $S \in \mathcal{I}_2$ i.e., $G \setminus S$ is connected. Therefore $\exists$ another spanning tree of G in $E \setminus S$. Hence there are atleast two edge disjoint spanning trees.

Question:4 *3 Matroid intersection Problem is NPC*

Proof. To prove this we show, directed Hamiltonian path $\leq_p$ 3 matroid intersection.
Directed Hamiltonian Path Problem: Given directed graph $D = (V, A)$ and $s, t \in V$ Is there a path from s to t that goes through each vertex exactly once.
$M_1 = (E, \mathcal{I}_1)$ is the graphic matroid of underlying undirected graph i.e., E is the same set of edges as A ignoring direction.
$M_2 = (A, \mathcal{I}_2)$, $\mathcal{I}_2 = \{X \in E \mid$ each $v \in V$ has at most one incoming edge in X, except s which has no edge entering it$\}$.
$M_3 = (A, \mathcal{I}_3)$, $\mathcal{I}_3 = \{Y \in E \mid$ each $v \in V$ has at most one outgoing edge in Y, except t which has no edge leaving it$\}$.

Now we prove that M_2 and M_3 are matroids. In question 6 we prove that M_2 is a matroid. For M_3 heriditary property holds trivially. Suppose $A, B \in \mathcal{I}_3$ and $|A| < |B|$ but $\nexists x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{I}_3$.
But each edge in B is outgoing edge for exactly one vertex and since no of edges in B is greater than no of edges in A then there is atleast one vertex, v with outdegree 0 in A but has outdegree

1 is B. Therefore we can add this edge comming out of v to A.

$M_1 \cap M_2 \cap M_3$ = set of vertex-disjoint directed paths with one starting at s and one ending at t, so there exists a Hamiltonian path iff max cardinality of intersection has size $|V| - 1$. □

Question:5 *We are given an undirected multigraph G(V,E) where each edge is colored by one of $k \leq n$. The Objective is to find a spanning tree T in G such that each edge in T has distinct colors.*

$M_1 = (E, \mathcal{I}_1)$ be the graphic matroid of G and

Let E_i be the egde set of color k_i, for $k_i \in [k]$, $E = \biguplus_{i=1}^{k} E_i$

$M_2 = (E, \mathcal{I}_2)$ be a partition matroid and

$$\mathcal{I}_2 = \{X \subseteq E \mid X \cap E_i = 1 i.e., \forall x_i \in X, x_i \text{ has different colors}\}$$

Claim: Set family M_2 forms a matroid

Proof. If a set of edges have different color then all its subset contains edges of different color, So heriditary property holds. Suppose $A, B \in \mathcal{I}_2$ and $|A| < |B|$. Since $B \in \mathcal{I}_2, |B| \leq k$ and all edges of B has different colors. Then there existes atleast one edge of color k_i in B such that there is no edge of color k_i in A. So A can be extended by that edge. Hence property (I3) is true for M_2. □

If M_1 and M_2 has $|V| - 1$ size intersection, then there is a spanning tree T in G such that each edge in T has distinct colors.

Question:6 *Given a directed graph, D(V,A), check whether D has arboresence*

Graph G is an arborescence[2] of root r, an r-arborescence, or just an arborescence iff:

1. Each vertex $v \neq r, v$ has indegree exactly 1.
2. indegree of r is 0.
3. For each $v \in V$ such that $v \neq r$ there is exactly one directed walk from r to v.

$M_1 = (E, \mathcal{I}_1)$ be the graphic matroid of underlying undirected graph. So E is the same set of edges as A ignoring direction.
$M_2 = (A, \mathcal{I}_2)$
$I_2 = \{X \subset A \mid$ if $(i_1, j_1) \in X$ and $(i_2, j_2) \in X$ and $j_1 \neq j_2\}$ i.e., no two edges in X are incident on the same vertex, i.e., the indegree of each vertex is atmost one.

Claim: M_2 is a matroid

Proof. Property (I1) and (I2) holds trivially.
Suppose (I3) does not hold. Then there is $|A|, |B| \in \mathcal{I}_2 . |A| < |B|$ then there is atleast one vertex,

v in B with indegree 1 that has indegree 0 in A. But then we can add the edge incident on vertex v to A i.e, A can be extended. So (I3) holds □

References

[1] Tutte, W.T. (1970), *Introduction to theory of Matroids*, American Elsevier.

[2] Tutte, W.T. (2001), *Graph Theory, Cambridge University Press*, ISBN 978-0-521-79489-3.