Advanced Graph Algorithms Jan-Apr 2014

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1 Overview

In the last lecture we have seen structures on which greedy algorithms are effective, and defined different kinds of Matroids, and representations for certain kinds of matroids. We then, described two important problems:

- *Matroid Intersection:* Given matroids $M_1 = (E, I_1), M_2 = (E, I_2), \ldots, M_n = (E, I_n)$ on the same ground set E, and a number k, does there exist a set $X \subseteq E$ such that $|X| \geq k$ and $X \in I_i$ for $1 \leq i \leq n$?
- *Matroid Pairity:* Given a Matroid $M = (E, I)$, with a pairing among the elements of the set E (hence, $|E|$ is even), and a number m, does there exist a set $X \subseteq E$ containing only the elements from some $t \geq m$ pairs in E, such that $X \in I$?

We have reduced a few important graph problems to these two problems. Further, we have seen that 3−Matroid Intersection is NP−Complete.

In this lecture we shall further reduce Matroid Intersection to Matroid Pairity and look into an efficient algorithm for Matroid Pairity. Also, in this lecture, whenever we say Matroid Intersection, we mean 2−Matroid Intersection.

2 Algorithms for Matroid Pairity and Matroid Intersection

To address any of the above problems we, first, assume that the matroids are representable and we are provided with a representations of them as linear matroids. We reduce both these problems to finding the rank of a matrix which is of size $O(e)$ where e is the size of the ground set. But first, we reduce Matroid Intersection to Matroid Pairity.

2.1 Reducing Matroid Intersection Problem to Matroid Pairity Problem

Theorem 1. *Matroid Intersection* $\leq_P$ *Matroid Pairity.*

Proof. We give a reduction for any Matroid Intersection problem $\langle M_1 = (E, I_1), M_2 = (E, I_2)\rangle$, where M_1 and M_2 are linear matroids of ranks r_1 and r_2 respectively, to a Matroid Pairity Problem $\langle M' = (E', I')\rangle$. We can assume, without loss of generality, that the order of the elements in E represented by columns in M_1 is the same as the order of the elements in E represented by the

columns in M_2. Otherwise, we can swap rows till this order is achieved.

Let $e_1^1, e_2^1, \ldots, e_t^1$ be the columns of the linear matroid M_1, and $e_1^2, e_2^2, \ldots, e_t^2$ be the columns of M_2. Construct a matroid M' of rank $r_1 + r_2$ as follows:

$$M' = \left[\begin{array}{c|c|c|c|c|c|c} e_1^1 & 0 & e_2^1 & 0 & \ldots & c_t^1 & 0 \\ 0 & e_1^2 & 0 & e_2^2 & \ldots & 0 & c_t^2 \end{array}\right]$$

Pair e_i^1 with e_i^2 for $1 \leq i \leq t$.

Call the columns of M' $f_1, f_2, f_3, \ldots, f_{2t}$. We show a bijection between a set pairs, of independent columns in M' and an intersection between M_1 and M_2. Let $f_{i_1}, f_{i_1+1}, f_{i_2}, fi_2 + 1, \ldots f_{i_m}, f_{i_m+1}$ be m pairs of rows, that are all independent such that each $i_k = 2j_k - 1$ for some integer j_k. Since $f_{i_1}, f_{i_2}, \ldots, f_{i_m}$ are all independent, $e_{k_1}^1, e_{k_2}^1, \ldots, e_{k_m}^1$ are all independent, since these columns have zero entries from the r_1+1^{th} to the r_2^{th} row, and the other columns have zeros in the first r_1 rows. By a similar argument, $e_{k_1}^2 e_{k_2}^2, \ldots, e_{k_m}^2$ are independent. So, $f_{i_1}, f_{i_1+1}, f_{i_2}, fi_2 + 1, \ldots f_{i_m}, f_{i_m+1}$ can be mapped to the two sets, $\{e_{k_1}^1, e_{k_2}^1, \ldots, e_{k_m}^1\}$ and $\{e_{k_1}^2 e_{k_2}^2, \ldots, e_{k_m}^2\}$, both which contain the same elements from E (since same numbered columns in M_1 and M_2 are mapped to the same element in E).

This mapping is a bijection because, if two sets of pairs in M' are mapped to the same sets in M_1 and M_2, then they must have the samen column indices (because that is how they are paired), and since there is a one to one mapping between the columns of M_1 and M' and M_2 and M', the two sets must be mapped to the same set of pairs in M'. □

It helps to describe some conventions that will be used in the rest of the notes. For any matrix A and sets X and Y which are subsets of the indices of the rows and columns respectively, $A[X, Y]$ denotes the sub-matrix of A induced on rows from X and columns from Y. Since most of the matrices we will be dealing with are square matrices, we use the same set of indices for the rows and columns. Also, since we will also be dealing with a lot of symmetric/skew-symmetric matrices, $A[X]$ is to be understood as $A[X, X]$. M represents a set of matchings over the indices of a square matrix. $M|_X$ is the set of matchings in a set $X \subseteq V$, where V is the set of indices of the matrix.

2.2 Reducing Matroid Intersection to Rank Finding

We can assume the matroids have the same rank because otherwise, we can add dummy rows to the linear matroid while preserving linear independence of columns.The Matroid Intersection problem is reduced to finding the rank of a matrix. Construct the matrix $Z_{(r+e)\times(r+e)}$ from the linear matroids M_1 and M_2 as follows:

$$Z = \left[\begin{array}{cc} 0 & M_1 \\ M_2^T & T \end{array}\right]$$

Where T is defined as follows:

$$T_{i,j} = \begin{cases} t_i & \text{if } (i = j) \\ 0 & \text{otherwise} \end{cases}$$

let $\mu(M_1, M_2)$ be the largest intersection between M_1 and M_2.

Theorem 2. $\mu(M_1, M_2) = rank(Z) - e$

This is a result due to [1].We shall not prove this here. However, we give the full algorithm for the Matroid Pairity.

2.3 Reducing Matroid Pairity to Rank Finding

First, we begin by proving the following useful lemma:

Lemma 3. *If K and Q are two skew symetric matrices of size $V \times V$ such that all the entries in Q are distinct indeterminates and no two non-zero elements of K and Q have the same indices, then $K + Q$ is non singular iff there is a partition of V into X and Y such that $K[X]$ and $Q[Y]$ are both, non-singular.*

Proof. Let $A = K + Q$. We shall use a, k, q to denote elements of matrices A, K, Q respectively. Recall from *Lecture 9* that $det(A) = (Pf(A))^2$. So, A is non-singular iff $Pf(A) \neq 0$.

Now, $Pf(A) = \sum_{\mu \in M} sgn(\mu) \prod a_\mu$. For every $I \subseteq V$, we can take one matching in I which is from K and one fron $V \setminus I$ which is from Q to give one non-zero matching of A (a matching μ for which $\prod a_\mu \neq 0$) since every non-zero element in A is either from K or from Q. Hence, $Pf(A) = \sum_{I \subseteq V} (\pm 1)(\sum_{\mu_1 \in M|_I} \prod k_{\mu_1})(\sum_{\mu_2 \in M|_{V \setminus I}} \prod q_{\mu_2}) = \sum_{I \subseteq V} (Pf(K[I]))(Pf(Q[V \setminus I]))$. Therefore, if $det(A) \neq 0$, then at least one term from the sum is non-zero, so there exists some $X \subseteq V$ such that $Pf(K[X])Pf(Q[V \setminus X])$ is non-zero, hence, putting $Y = V \setminus X$, we say, since $Pf(K[X])$ and $Pf(Q[Y])$ are both non-zero, $K[X]$ and $Q[Y]$ are both, non-singular.

The other direction of the proof comes from observing that since all the indeterminates of Q are distinct, no two non-singular sub-matrices of Q are polynomials of degree equal to the size of the correponding sub-matrices on the same set of indeterminates, hence if $det(A) = 0$, all the terms in the sum must be equal to zero, since they cannot cancel each other out. □

Lemma 4. *Let A be a skew symetric $n \times n$ matrix, $P = \{1, 2, \ldots, p\}$ and $Q = \{p+1, p+2, \ldots, n\}$.If $A[P, P] = 0$, $A[Q, Q] = 0$ and A is non-singular, then $|P| = |Q|$*

Proof. if $|P| \neq |Q|$, then we can show that by finding determinants via multiplication of block determinants, that A is singular.

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where W, X, Y, Z are all square matrices of the same dimension (since A is a skew symmetric non-singular matrix, it must be of even size. If $|P| < |Q|$, then $Z = 0$ and X and Y have zero rows and zero columns respectively. If $|P| > |Q|$, then $W = 0$ and X and Y have zero columns and zero rows respectively. Hence $|P| = |Q|$ □

Now, we are ready to proceed with the algorithm. The first thing we need to impose is that the columns that are paired in A are adjacent to each other, i.e for every k, the $2k - 1^{th}$ columns is paired with the $2k^{th}$ column. Then, we construct a matrix $Z_{(r+2e)\times(r+2e)}$ fom the linear matroid $A_{r\times 2e}$ as shown below:

$$C = \begin{bmatrix} 0 & A \\ -A^T & T \end{bmatrix}$$

where T is a $2e \times 2e$ skew-symmetric matrix defined as:

$$T_{i,j} = \begin{cases} t_k & \text{if } (i,j) = (2k-1, 2k) \text{ for some integer } k \\ -t_k & \text{if } (i,j) = (2k, 2k-1) \text{ for some integer } k \\ 0 & \text{otherwise} \end{cases}$$

Theorem 5. *Let μ be the maximum size of pairs that are independent in $A_{r\times 2e}$. Then, $2\mu = rank(Z) - 2e$.*

Proof. We shall show the following two inequalities

1. $2\mu \geq rank(Z) - 2e$

2. $2\mu \leq rank(Z) - 2e$

Note that the matrix T is non singular because $det(T) = \prod t_i^2$. So we can add more rows and columns to it to make it a skew symmetric square matric of size $rank(Z)$ (follows from extension property of linear matroids, and *Lemma* from *Lecture 11*). Call this matrix Z'. Call the indices of Z' which corespond to matrix T W and the remaining R'. Write Z' as the sum of two matrices K and Q, where:

$$K = \begin{bmatrix} 0 & A[R', W] \\ -A[R', W]^T & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$$

Because Z' is non singular by *Lemma 3*, we can find a partition X, Y of $R' \cup W$ such that $K[X]$ and $Q[Y]$ are both non-singular. $Y \subset W$ because otherwise, $Q[Y]$ has zero rows (and columns). Also, note that in T, for every t, the only non-zero entry of the $2t - 1^{th}$ row is in the $2t^{th}$ row and

the only non-zero entry of the $2t^{th}$ row is in the $2t - q^{th}$ row, so, if Y must have both the $2t^{th}$, and the $2t - 1^{th}$ row or neither of them. Hence Y represents a set of pairs in A. So, $W \setminus Y$ also represents a set of pairs in A. Also, $X = R' \cup (W \setminus Y)$.

By *Lemma 4*, $|R'| = |W \setminus Y|$. So, $det(K'[X]) = -det(A[R', W \setminus Y]) \times det(-A[R', W \setminus Y]^T) = det(A[R', W \setminus Y])^2$(block determinants). So, the columns of $A[R', W \setminus Y]$ are linearly independent since it is a full rank matrix. Hence the columns $W \setminus Y$ are independent in A and are in pairs.

Hence, $2\mu \geq |W \setminus Y| = |R'| = rank(Z) - 2m$, thus proving the first part of the theorem.

Consider the set X' of columns in A that correspond to the maximum pairity matching. There must exist a set of rows X'' such that $A[X'', X']$ has independent rows. Such a set must exist because $rank(A[*, X']) = |X'|$). now, consider the matrix $z[x'' \cup w]$ and write it as the sum of two matrices p' and s' as follows

$$P' = \begin{bmatrix} 0 & A[X', W] \\ -A[X', W]^T & 0 \end{bmatrix}$$

$$S' = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$$

And similar to the previous case, we partition the row indices as $X = X' \cup X''$ and $Y = W \setminus X'$. Since X' consists of pairs corresponding to the linear matriod A, it has pairs corresponding to the matrix T, and hence $W \setminus X'$. Hence, $det(S'[Y]) \not\equiv 0$. By *Lemma 3*, $Z[X' \cup W]$ is non-singular. Hence, $rank(Z) \geq |X'| + 2m = 2\mu + 2m$, or $2\mu \leq rank(Z) - 2m$, proving the next part of the theorem and completing the proof.

Note that while proving *Lemma 3*, we have assumed that Q has distinct variables, but here, T does not have distinct variables. However, the variables in the determinants of each sub-matrix with a non-zero determinant are different. This is the only fact that we have used in proving the lemma, so this argument is still valid. □

Hence, we have reduced the problem of Matroid Pairity to that of finding the rank of a mixed matrix. To find out the rank, we simply substitute values of the indeterminates of a sufficiently large field and can get the correct rank with a bound probability on error. This can be done very efficiently in randomized e^{ω} time

References

[1] Nicholas J. A. Harvey: Algebraic Algorithms for Matching and Matroid Problems. SIAM J. Comput. 39(2): 679-702 (2009)

[2] Ho Yee Cheung, Lap Chi Lau, Kai Man Leung: Algebraic Algorithms for Linear Matroid Parity Problems. SODA 2011: 1366-1382