Q Suppose $a, b \in \mathbb{R}$. Consider the following system of eqns:

$$2x_1 + x_2 + ax_3 = 1, \quad x_1 + x_3 = 2, \quad x_1 + x_2 - x_3 = b$$

a) Apply suitable elementary row operations on the augmented matrix to find an equivalent system such that its coefficient matrix is RRE for every a & b.

b) Using part (a) find all possible $a, b \in \mathbb{R}$ s.t. the system has i) no solutions ii) a unique solution, iii) infinitely many solutions.

c) Write all the solutions when the system is consistent.

Solution: The augment matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & a & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & b \end{array}\right) \overset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & a & 1 \\ 1 & 1 & -1 & b \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 1 & -2 & b-2 \end{array}\right)$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 0 & -a & b+1 \end{array}\right).$$ If $a = 0$, the coefficient matrix is RRE

uppose $a \neq 0$. Apply $R_3 \to -a^{-1}R_3$, $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 + \frac{b+1}{a} \\ 0 & 1 & 0 & -3 + \frac{(b+1)(a-2)}{a} \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2a+b+1}{a} \\ 0 & 1 & 0 & \frac{ab-2a-2b-2}{a} \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right).$$ where the

oefficient matrix is RRE.

Thus if $a \neq 0$, the system has a unique solution & it is

$$\left(\frac{2a+b+1}{a}, \frac{ab-2(a+b+1)}{a}, -\frac{b+1}{a}\right).$$

$a = 0$ but $b+1 \neq 0$ i.e. $b \neq -1$, then the system is inconsistent.

$a = 0$ & $b = -1$, the system is $x_1 + x_3 = 2$, $x_2 - 2x_3 = -3$

here x_3 is the free unknown. Thus the (infinite number of) olutions are (by putting $x_3 = \lambda$) $(2-\lambda, -3+2\lambda, \lambda)$ for every $\lambda \in \mathbb{R}$.