Q Suppose u, v, w are vectors in an arbitrary Vector space V over a field $\mathbb{F}$. Show that $\{u, v, w\}$ is linearly independent if & only if $\{u+v, v+w, w+u\}$ is linearly independent.

<u>Solution:</u> "if" Suppose $\{u+v, v+w, w+u\}$ is linearly independent. We will show that $\{u, v, w\}$ is linearly independent.

Let $w' = u+v, \quad u' = v+w, \quad v' = w+u$. Then $\{u', v', w'\}$ is linearly independent & $x w' + y u' + z v' = (x+z) u + (x+y) v + (y+z) w$.

Suppose $a, b \in \mathbb{R}$ ~~Then~~ such that $au + bv + cw = 0$

We need to show that $a = b = c = 0$.

First we find x, y, z such that

$$x + z = a, \quad x + y = b, \quad y + z = c \longrightarrow (*)$$

Augmented matrix is $\begin{pmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 1 & 1 & c \end{pmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c-b+a \end{pmatrix} \overset{R_3 \to \frac{1}{2} R_3}{\sim} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{pmatrix}$

$R_1 \to R_1 - R_3$

$\sim$

$R_2 \to R_2 + R_3$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a - \frac{c-b+a}{2} \\ 0 & 1 & 0 & b-a+\frac{c-b+a}{2} \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a+b-c}{2} \\ 0 & 1 & 0 & \frac{-a+b+c}{2} \\ 0 & 0 & 1 & \frac{a-b+c}{2} \end{array}\right)$

Thus $x = \frac{a+b-c}{2}, \quad y = \frac{-a+b+c}{2}, \quad z = \frac{a-b+c}{2}$

If we make these choices of x, y & z then

$(x+z) u + (x+y) v + (y+z) w = 0$

$\Rightarrow$ ~~$a w' + bu$~~

$\Rightarrow x w' + y u' + z v' = 0 \implies x = y = z = 0$ ($\because \{u', v', w'\}$ is linearly independent)

Then $a = 0, b = 0, c = 0$ (since x, y, z satisfy $(*)$)

"Only if" part is easier & left to the students.