12Q. Let $W_1 = \{(x,y,z,w) : 2x+y=0,\ x+y+z=0\}$

$W_2 = \{(x,y,z,w) : x+y+z+w=0,\ x+2y+3z+w=0\}$

Find a basis of $W_1 \cap W_2$, a basis of $W_1 + W_2$ and write W_1+W_2 as the solution space of a system of ~~q~~ homogeneous linear equations.

Solution: $W_1 \cap W_2 = \{(x,y,z,w) : 2x+y=0,\ x+y+z=0,\ x+y+z+w=0,\ x+2y+3z+w=0\}$.

Consider the coefficient matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \underset{\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}}{\overset{R_1 \leftrightarrow R_2}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \overset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\underset{R_2 \to (-1)R_2}{\overset{\substack{R_1 \to R_1 + R_2 \\ R_4 \to R_4 + R_2}}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ which is RRE. Hence,

$W_1 \cap W_2 = \{(x,y,z) : x - z = 0,\ y + 2z = 0,\ w = 0\}$

~~and~~ $= \{(\lambda, -2\lambda, \lambda, 0) : \lambda \in \mathbb{R}\}$ (since z is the only free unknown is the above system set $z = \lambda$)

Thus $\{(1,-2,1,0)\}$ is a basis of $W_1 \cap W_2$.

Rank of $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 2$ & rank of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = 2$

so $\dim W_1 = 4-2 = 2$. & $\dim W_2 = 4-2 = 2$.

To get a basis of W_i by extending $\{(1,-2,1,0)\}$ we need one more vector for each $i = 1,2$.

$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ &

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$,

~~(1, -2, 1, 0)~~

$e_4 \in W_1$ & $\{(1,-2,1,0), e_4\}$ is linearly ind.

$(-1,0,0,1) \in W_2$ & $\{(1,-2,1,0), (-1,0,0,1)\}$ is linearly ind.

$\{(1,-2,1,0), (0,0,0,1), (-1,0,0,1)\}$ is a basis of $W_1 + W_2$