Solution of 12b continued:

We have seen that $\{(1,-2,1,0), (0,0,0,1), (-1,0,0,1)\}$ is a basis of W_1+W_2. To write W_1+W_2 as the solution space of a hom system of linear eqns

Consider
$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

which is row equivalent to
$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
by $R_3 \to (-1)R_3$, $R_3 \leftrightarrow R_1$, $R_1 \leftrightarrow R_2$

$$\overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{R_2 \to R_2(-\frac{1}{2})}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $W_1 + W_2 =$ Span of $\{(1,0,0,-1), (0,1,-\frac{1}{2},-\frac{1}{2}), e_4\}$

$$= \left\{\left(a, b, -\frac{b}{2}, -a-\frac{b}{2}+c\right) : a, b, c \in \mathbb{R}\right\}$$

$$= \left\{(x,y,z,w) : z = -\frac{y}{2}\right\}$$ ~~$w = -x - \frac{y}{2}$~~ (the fourth component can be any real number)

$$= \{(x,y,z,w) : y + 2z = 0\}$$

$$= \{(\lambda, -2\mu, \mu, \delta) : \lambda, \mu, \delta \in \mathbb{R}\}$$ (note that x, z, w are free unknowns)

Thus $\{(1,0,0,0), (0,-2,1,0), (0,0,0,1)\}$ is also a basis → not asked.