Q13. Write the following space as the solution space of a system of homogeneous linear equations.

$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \right\}$ as a subspace of $M_{2\times 2}(\mathbb{R})$.

<u>Solution:</u> Consider the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ whose first rows is obtained from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ & the second from $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, (remember how you got)

Now,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \overset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \end{pmatrix}$$

$$\overset{R_2 \to -\frac{1}{3} R_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \end{pmatrix} \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

~~Thus $W = \{ a(1, 0, \frac{5}{3}, \frac{2}{3}) + b(0, 1, \frac{2}{3}, \frac{5}{3}$~~

Thus $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{5}{3} \end{pmatrix} \right\}$ (see how we got these matrices from the rows)

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{5}{3} \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ \frac{5a+2b}{3} & \frac{2a+5b}{3} \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) ; z = \frac{5x+2y}{3}, \ w = \frac{2x+5y}{3} \right\}$$

$$= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : 5x + 2y - 3z = 0, \ 2x + 5y - 3w = 0 \right\}$$