Q14 Show that the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & \frac{9}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{6}{5} \end{pmatrix}$ is diagonalizable.

Find an invertible matrix P s.t. $P^{-1}AP$ is diagonal.

<u>Soln:</u> The characteristic polynomial is

$$\det\begin{pmatrix} x-1 & 0 & 0 \\ \frac{2}{5} & x-\frac{9}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & x-\frac{6}{5} \end{pmatrix} = (x-1)^2(x-2).$$ The eigen values are 1 & 2.

To find the eigen space corresponding to 1:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ -\frac{2}{5}x+\frac{9}{5}y+\frac{2}{5}z \\ -\frac{x}{5}+\frac{2}{5}y+\frac{6}{5}z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{array}{l} x=x \\ -2x+4y+2z=0,\ -x+2y+z=0 \end{array}$$

$\Leftrightarrow -x+2y+z=0.$ The coefft matrix of the system $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The solution space is ~~one~~ two dimensional (since the rank of the coefft matrix is one). &

$(2,1,0)$ (putting $z=0$, $y=1$) &

$(1,0,1)$ (putting $y=0$ $z=1$)

are linearly independent eigen vectors. Thus $\{(2,1,0),(1,0,1)\}$ is a basis of the eigenspace corresponding to 1.

To find the eigen space corresponding to 2:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{array}{l} x=2x \\ -2x-y+2z=0 \\ -x+2y-4z=0 \end{array} \Leftrightarrow \left.\begin{array}{l} x=0 \\ x-2y-4z=0 \\ 2x+y-2z=0. \end{array}\right\} \Rightarrow \begin{array}{l} x=0 \\ y-2z=0 \end{array}$$

The coefft matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & -4 \\ 2 & +1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ which is of rank 2.

The dimension of the soln space is one & $(0,2,1)$ is an eigen vector. Thus $\{(0,2,1)\}$ is a basis of the eigen space corresponding to 2.

If $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ then $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(See how to get P from the bases of the eigen spaces.