Q.15 Let ~~V~~ $P_3 = \{a+bx+cx^2 : a,b,c \in \mathbb{R}\}$.

Define $\langle f|g\rangle = \int_0^1 f(x)g(x)\,dx$.

Find an orthogonal basis of P_3.

Solution: ~~Obser~~ Recall $\{1, x, x^2\}$ is a basis of P_3.

Use Gram-Schmidt process:

$$v_1 = 1,\quad v_2 = x - \frac{\langle x|1\rangle}{\|1\|^2}1. = x - \int_0^1 t\,dt \Big/ \left(\int_0^1 1\,dt\right)^2 = x - \frac12$$

$$v_3 = x^2 - \frac{\langle x^2|1\rangle}{\|1\|^2}1. - \frac{\langle x^2|x-\frac12\rangle}{\|x-\frac12\|^2}(x-\tfrac12)$$

$$= x^2 - \int_0^1 t^2\,dt - \frac{\int_0^1 t^2(t-\frac12)\,dt}{\int_0^1 (t-\frac12)^2\,dt}\cdot(x-\tfrac12)$$

$$= x^2 - \frac13 - \frac{\left(\frac14 - \frac12\cdot\frac13\right)}{\left(\frac13 - \frac12 + \frac14\right)}(x-\tfrac12)$$

$$= x^2 - \frac13 - \frac{3-2}{12}\cdot\frac{12}{(4-6+3)}(x-\tfrac12)$$

$$= x^2 - \frac13 - 1\,(x-\tfrac12)$$

$$= x^2 - x + \frac16.$$ Thus $\{1,\ x-\frac12,\ x^2-x+\frac16\}$ is an orthogonal basis of P_3

Not a part of the solution.

Checks: $\langle 1|x-\frac12\rangle = \int_0^1 (t-\frac12)\,dt = \frac{t^2}{2}\Big|_0^1 - \frac12 = 0$

- $\langle 1|x^2-x+\frac16\rangle = \int_0^1 t^2 - t - \frac16 = \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{6}\Big|_0^1 = \frac13 - \frac12 + \frac16 = 0$
- $\langle (x-\frac12)|(x^2-x+\frac16)\rangle = \int_0^1 (t-\frac12)(t^2-t+\frac16)\,dt = \int_0^1 t^3 - t^2 + \frac{t}{6} - \frac{t^2}{2} + \frac{t}{2} - \frac1{12}\,dt$

$$= \int_0^1 t^3 - \frac32 t^2 + \frac23 t - \frac1{12}\,dt = \frac{t^4}{4} - \frac32\cdot\frac13 t^3 + \frac23\cdot\frac12 t^2 - \frac1{12}t\Big|_0^1$$

$$= \frac14 - \frac12 + \frac13 - \frac1{12} = \frac{3-6+4-1}{12} = 0.$$