Q.16 Show that the Lipschitz condⁿ is satisfied by the function $f = |\sin y| + x$ at every point on the xy-plane though its partial derivative with respect to y does not exist on the line $y=0$.

Solⁿ: $\lim\limits_{t \to 0} \frac{|\sin t| - |\sin 0|}{t} = \lim\limits_{t \to 0} \frac{|\sin t|}{t}$ does not exist, so $\frac{d(|\sin y|)}{dy}$ does not exist at $y=0$.

$$\frac{|f(y_2, x) - f(y_1, x)|}{|y_2 - y_1|} = \frac{|(|\sin y_2| + x) - (|\sin y_1| + x)|}{|y_2 - y_1|}$$

$$= \frac{||\sin y_2| - |\sin y_1||}{|y_2 - y_1|} \leq \frac{|\sin y_2 - \sin y_1|}{|y_2 - y_1|}$$

$$= \frac{2\left|\sin \frac{y_2 - y_1}{2}\right| \left|\cos \frac{y_2 - y_1}{2}\right|}{|y_2 - y_1|}$$

Using $\left|\frac{\sin(x)}{x}\right| < 1$ we have

$$\frac{|f(y_2, x) - f(y_1, x)|}{|y_2 - y_1|} < \left|\cos\left(\frac{y_2 - y_1}{2}\right)\right| \leq 1.$$

Thus $|f(y_2, x) - f(y_1, x)| \leq |y_2 - y_1|$