Problem: Suppose W_1 and W_2 are subspaces of a vector space V over a field $\mathbb{F}$. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Solution: $(\Leftarrow)$ $W_1 \subset W_2 \Rightarrow W_1 \cup W_2 = W_2$, which is a subspace of V

$W_2 \subset W_1 \Rightarrow W_1 \cup W_2 = W_1$, which is a subspace of V.

So, in either case $W_1 \cup W_2$ is a subspace of V.

$(\Rightarrow)$ Given: $W_1 \cup W_2$ is a subspace of V.

To show: either $W_1 \subset W_2$ or $W_2 \subset W_1$.

We'll prove this by contradiction.

Suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$.

Then there exist vectors w_1 and w_2 such that

$$\left.\begin{array}{l} w_1 \in W_1 \ , \ w_1 \notin W_2 \\ \& \ w_2 \in W_2 \ , \ w_2 \notin W_1 \end{array}\right\} (*)$$

So, both w_1 and w_2 belong to $W_1 \cup W_2$.

Claim: $w_1 + w_2 \notin W_1 \cup W_2$

$w_1 + w_2 \in W_1 \Rightarrow w_2 = (w_1 + w_2) - w_1 \in W_1$ ($\because W_1$ is a subspace)

Illy, $w_1 + w_2 \in W_2 \Rightarrow w_1 = (w_1 + w_2) - w_2 \in W_2$ ($\because W_2$ is a subspace)

So, we arrive at a contradiction to $(*)$ if we assume $w_1 + w_2 \in W_1 \cup W_2$.

Hence, $w_1 + w_2 \notin W_1 \cup W_2$ which implies that $W_1 \cup W_2$ is not a subspace as $w_1, w_2 \in W_1 \cup W_2$.

This completes the proof of the "only if" part.

WARNING: Just providing an example where $W_1 \not\subset W_2$, $W_2 \not\subset W_1$ and $W_1 \cup W_2$ is not a subspace, is <u>NOT</u> a correct proof.