Problem: Let $V = M_n(\mathbb{R})$, $W_1 = \{A \in V : A^t = A\}$

and $W_2 = \{A \in V : A^t = -A\}$

Prove that $V = W_1 \oplus W_2$.

Soln: Note that W_1 and W_2 are subspaces of V (Prove it!)

Further let $A \in M_n(\mathbb{R})$.

Write $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$

Then $B^t = B$ and $C^t = -C$

So, $B \in W_1$ & $C \in W_2$

Also, $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$

$= B + C \in W_1 + W_2$.

This proves $V = W_1 + W_2$.

To show $V = W_1 \oplus W_2$, we need $W_1 \cap W_2 = \{0\}$.

Suppose $A \in W_1 \cap W_2$

$\Rightarrow$ $A^t = A$ and $A^t = -A$

$\Rightarrow$ $A = -A$ $\Rightarrow$ $2A = 0$ $\Rightarrow$ $A = 0$ (the zero matrix)

$\therefore$ $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$.