5Q Show that if ~~If~~ $u_1 = (x_1, x_2, x_3)$, $u_2 = (y_1, y_2, y_3)$, $u_3 = (z_1, z_2, z_3)$ ~~are~~ such that $B = \{u_1, u_2, u_3\}$ is linearly independent subset of $\mathbb{R}^3$, then ~~any vector $v = (\alpha, \beta, \gamma)$~~, B is a basis of $\mathbb{R}^3$.

Solution: Since B is given to be linearly independent, we have to show that every vector $v = (\alpha, \beta, \gamma)$ can be written as a linear combination of vectors in B (i.e. in Span (B)), that is to find scalars $a, b, c \in \mathbb{R}$ s.t.

$$(\alpha, \beta, \gamma) = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) + c(z_1, z_2, z_3)$$

$$\Leftrightarrow (\alpha, \beta, \gamma) = (ax_1 + by_1 + cz_1, ax_2 + by_2 + cz_2, ax_3 + by_3 + cz_3)$$

$$\Leftrightarrow ax_1 + by_1 + cz_1 = \alpha, \quad ax_2 + by_2 + cz_2 = \beta, \quad ax_3 + by_3 + cz_3 = \gamma$$

$$\Leftrightarrow \begin{aligned} x_1 a + y_1 b + z_1 c &= \alpha \\ x_2 a + y_2 b + z_2 c &= \beta \\ x_3 a + y_3 b + z_3 c &= \gamma \end{aligned}$$

> Remark: Finding the coeffients a, b, c boils down to solving a system of non homogeneous linear equations.

Which has a solution if and only if

$$\text{Rank}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \text{Rank}\begin{pmatrix} x_1 & y_1 & z_1 & \alpha \\ x_2 & y_2 & z_2 & \beta \\ x_3 & y_3 & z_3 & \gamma \end{pmatrix} \quad \cdots\cdots (*)$$

Since B is linearly independent, $\text{Rank}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 3$

& Rank (of the augmented matrix) is at least the rank of the coeffcin't matrix and at most 3, ~~the equati~~ the relation $(*)$ always holds. Therefore there exist $a, b, c \in \mathbb{R}$ s.t. $v = au_1 + bu_2 + cu_3$.