6Q. Prove the following ~~statement~~ statement:

A map $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear tranformation if and only if $T(x,y) = (ax+by, cx+dy)$ for some $a, b \in \mathbb{R}$.

Solution: Suppose $T(x,y) = (ax+by, cx+dy)$. We will show that T is a linear trawsformation, that is,

$$T(p(x_1,y_1) + q(x_2,y_2)) = p\,T(x_1,y_1) + q\,T(x_2,y_2)$$

for any $p, q \in \mathbb{R}$ & $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$.

$$\begin{aligned}
LHS &= T(px_1+qx_2, py_1+qy_2) = \left(a(px_1+qx_2)+b(py_1+qy_2), c(px_1+qx_2)+d(py_1+qy_2)\right) \\
&= \left(p(ax_1+by_1)+q(ax_2+by_2), p(cx_1+dy_1)+q(cx_2+dy_2)\right) \\
&= p(ax_1+by_1, cx_1+dy_1) + q(ax_2+by_2, cx_2+dy_2) \\
&= RHS.
\end{aligned}$$

Next, Suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation (where $e_1=(1,0), e_2=(0,1)$)

$$\begin{aligned}
\text{Then} \quad T(x,y) &= T(xe_1+ye_2) \\
&= xT(e_1) + yT(e_2)
\end{aligned}$$

Let $T(e_1) = (\alpha, \beta)$, $T(e_2) = (\gamma, \delta) \in \mathbb{R}^2$

Then $T(x,y) = x(\alpha,\beta) + y(\gamma,\delta) = (\alpha x+\gamma y, \beta x+\delta y)$.

which are wanted to prove.

7Q. "The map $T: P_n \to P_{n+1}$ defined by $T(p)(x) = xp(x) + p(1)$ is a linear tranformation." Prove this statement.

(Here $P_n := \{p \in \mathbb{F}[x] : \deg(p) < n\}$.

Solution: ~~T(a~~ We have to show $T(ap+bq) = aT(p)+bT(q)$ ~~where~~ for $p, q \in$ ~~F~~ P_n & $a, b \in \mathbb{F}$. By definition

$(ap+bq)(x) = ap(x)+bq(x)$. ~~So~~ & ~~$T(ap+bq)(x) = x(ap+bq)(x)+$~~

$$\begin{aligned}
T(ap+bq)(x) &= x(ap+bq)(x) + (ap+bq)(1) = x(ap(x)+bq(x)) + ap(1)+bq(1) \\
&= a\{xp(x)+p(1)\} + b\{xq(x)+q(1)\} = aT(p)(x) + bT(q)(x).
\end{aligned}$$

Thus $T(ap+bq)(x) = \{aT(p)+bT(q)\}(x)$.