8 Q. Suppose $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear transformations. Show that $T_2 \circ T_1 : V_1 \to V_3$ is a linear transformation.

<u>Solution:</u> We have to ~~verify~~ show for $a, b \in \mathbb{F}$, $u, v \in V_1$,

$$(T_2 \circ T_1)(au + bv) = a\,(T_2 \circ T_1)(u) + b\,(T_2 \circ T_1)(v)$$

that is,

$$T_2\left(T_1(au+bv)\right) = a\,T_2(T_1(u)) + b\,T_2(T_1(v)) \text{ ------ } (*)$$

$$\cancel{\Leftrightarrow T_2\left(aT_1(u) + bT_1(v)\right)}$$

$$\text{LHS} = T_2\left(aT_1(u) + bT_1(v)\right) = a\,T_2(T_1(u)) + b\,T_2(T_1(v))$$

(since T_2 is a linear transformation)

$$= \text{RHS}.$$

<u>Thus $(*)$ is verified.</u>

9 Q. Show that if $T : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation, ~~show that~~ then $T = a_1 p_1 + a_2 p_2 + \cdots + a_n p_n$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and p_i is the map which maps $(x_1, x_2, \ldots, x_n)$ to x_i.

<u>Solution:</u> $T(x_1, x_2, \ldots, x_n) = T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$ (where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$).

$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \quad (\because T \text{ is linear})$$

$T(e_1), T(e_2), \ldots, T(e_n) \in \mathbb{R}$, Suppose $T(e_i) = b_i$.

$$\begin{aligned}
\text{Then}\quad T(x_1, x_2, \ldots, x_n) &= x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \\
&= b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \\
&= b_1 p_1(x_1, \ldots, x_n) + b_2 p_2(x_1, x_2, \ldots, x_n) + \cdots + b_n p_n(x_1, x_2, \ldots, x_n). \\
&= (b_1 p_1 + b_2 p_2 + \cdots + b_n p_n)(x_1, x_2, \ldots, x_n)
\end{aligned}$$

Thus $T = b_1 p_1 + b_2 p_2 + \cdots + b_n p_n$ with $b_1, b_2, \ldots, b_n \in \mathbb{R}$.