10Q. Suppose $S : \mathbb{R}^m \to \mathbb{R}^n$ is a map and $p_i : \mathbb{R}^n \to \mathbb{R}$ is the ith projection (i.e. $p_i(x_1, x_2, \dots x_n) = x_i$ for each $i = 1, 2, \dots, n$). Show that $S$ is a linear transformation if and only if $p_i \circ S$ is linear for each $i = 1, 2, \dots, n$.

<u>Solution:</u> First we show that $p_i : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation for each $1 \le i \le n$, that is,

$$p_i(a(x_1, x_2, \dots, x_n) + b(y_1, y_2, \dots, y_n)) = a\,p_i(x_1, x_2, \dots, x_n) + b\,p_i(y_1, y_2, \dots y_n).$$

for any $a, b \in \mathbb{R}$ & $(x_1, \dots, x_n); (y_1, \dots, y_n) \in \mathbb{R}^n$.

$$\begin{aligned} LHS &= p_i(ax_1 + by_1, ax_2 + by_2, \dots, ax_n + by_n) \\ &= \cancel{p}\ ax_i + by_i \quad (\text{by definition of } p_i) \\ &= a\,p_i(x_1, x_2, \dots, x_n) + b\,p_i(y_1, y_2, \dots, y_n). \end{aligned}$$

One can show that composition of two linear transformations is a linear transformation (see QWA7, Q8). Therefore, if $S$ is linear, so is $p_i \circ S$ for each $i$ (since, ~~each~~ $p_i$ is a linear transformation for each $i$).

Next, suppose ~~each~~ $p_i \circ S$ is a linear transformation for each $i$.

By definition of $p_1, p_2, \dots, p_n$, for any vector $v \in \mathbb{R}^n$

$$v = (p_1(v), p_2(v), \dots, p_n(v))$$

~~Since~~ Let $a, b \in \mathbb{R}$, $u, w \in \mathbb{R}^m$. Then $S(au + bw) \in \mathbb{R}^n$.

Hence
$$\begin{aligned} S(au + bw) &= (p_1(S(au + bw)), p_2(S(au + bw)), \dots, p_n(S(au + bw))) \\ &= ((p_1 \circ S)(au + bw), (p_2 \circ S)(au + bw), \dots, (p_n \circ S)(au + bw)) \end{aligned}$$

Observe, $\because p_i \circ S(au + bw) = a(p_i \circ S)(u) + b(p_i \circ S)(w)$,

$$\begin{aligned} S(au + bw) &= (a(p_1 \circ S)(u), a(p_2 \circ S)(u), \dots, a(p_n \circ S)(u)) + (b(p_1 \circ S)(w), b(p_2 \circ S)(w), \dots, b(p_n \circ S)(w)) \\ &= a(p_1(S(u)), p_2(S(u)), \dots, p_n(S(u))) + b(p_1(S(w)), p_2(S(w)), \dots, p_n(S(w))) \\ &= a\,S(u) + b\,S(w) \text{ which we wanted to show.} \end{aligned}$$