

Q. Solve the following system.

$$y_1' = 3y_1 - 4y_2$$

$$y_2' = y_1 - y_2$$

Soln: This can be written as

$$y' = Ay, \text{ with } A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

• Eigenvalues of A:

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 3 & 4 \\ -1 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 1) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1.$$

• Eigenvectors for $\lambda = 1$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -u_1 + 2u_2 = 0 \\ u_1 = 2u_2 \end{cases}$$

So, $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector & eigenspace is one dimensional.

$\therefore y^{(1)} = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a soln.

To find a second soln. L.I. to this, we need to solve ~~$(\lambda I + A)v = u$~~ $(A - \lambda I)v = u$

$$\text{i.e. } \begin{pmatrix} +2 & -4 \\ +1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow v_1 - 2v_2 = 1$$

Choosing $v_2 = 0$, we get $v = \begin{pmatrix} +1 \\ 0 \end{pmatrix}$

So, $y^{(2)} = t e^{\lambda t} u + e^{\lambda t} v$ with $\lambda = 1$

$$= t e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} +1 \\ 0 \end{pmatrix}$$

$\therefore$ The general soln. is $y = c_1 y^{(1)} + c_2 y^{(2)}$

$$\text{i.e. } y = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \left[t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} +1 \\ 0 \end{pmatrix} \right]$$