Q. Use Frobenius method to find a basis of solutions.

$(x^2+x)y'' + (4x+2)y' + 2y = 0$

Soln. Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$ and substituting y, y', y'' in the ODE, we get

$$(x^2+x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + (4x+2)\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + 2\sum_{m=0}^{\infty} a_m x^{m+r} = 0 \quad -(*)$$

The lowest power is x^{r-1} and equating the coeff. of x^{r-1} to zero gives the indicial eqn.

$r(r-1)a_0 + 2r a_0 = 0$

$\Rightarrow \quad r^2 + r = 0 \quad \Rightarrow \quad r = 0, -1$

The roots differ by an integer.

We have $r_1 = 0$ & $r_2 = -1$ (Recall r_1 is the larger root)

First soln: From $(*)$ with $r = r_1 = 0$ we have

$$m(m-1)a_m + (m+1)m a_{m+1} + 4m a_m + 2(m+1)a_{m+1} + 2a_m = 0, \quad m = 0, 1, 2, \ldots$$

This gives $a_{m+1} = -a_m$, $m = 0, 1, 2, \ldots$

Choosing $a_0 = 1$, $a_m = (-1)^m$

and $\boxed{y_1 = \sum_{m=0}^{\infty}(-1)^m x^m = \frac{1}{1+x}}$, $|x| < 1$

Second soln: Apply reduction of order method, $y_2 = u y_1$

After substituting & simplifying, we get

$x u'' + 2u' = 0 \quad \Rightarrow \quad \frac{u''}{u'} = -\frac{2}{x}$

This gives $u' = \frac{1}{x^2}$, so $u = -\frac{1}{x}$

$\therefore \quad y_2 = \frac{-1}{x(x+1)} = -\frac{1}{x} + \frac{1}{x+1}$

Since $\frac{1}{x+1} = y_1$ is a soln., we may take $\boxed{y_2 = \frac{1}{x}}$ as the 2nd soln. for the basis.

Note that y_1 & y_2 are clearly linearly independent.