Q. Find the eigenvalues & eigenfunctions of the following Sturm-Liouville problem.

$$y'' + \lambda y = 0 \quad , \; y(0) = y(2\pi) \; , \; y'(0) = y'(2\pi) .$$

Soln: For $\lambda < 0$, let $\lambda = -\mu^2$, $\mu \neq 0$.

Then $y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$y(0) = y(2\pi) \Rightarrow c_1 + c_2 = c_1 e^{2\pi\mu} + c_2 e^{-2\pi\mu}$

$y'(0) = y'(2\pi) \Rightarrow \mu c_1 - \mu c_2 = \mu c_1 e^{2\pi\mu} - \mu c_2 e^{-2\pi\mu}$

The only soln. is $c_1 = c_2 = 0$.

So, $y \equiv 0$. Thus there are $\boxed{\text{no negative eigenvalues.}}$

For $\lambda = 0$, the general soln. is

$$y = c_1 + c_2 x$$

$y(0) = y(2\pi) \Rightarrow c_2 = 0$.

$y = c_1$ satisfies the 2nd cond. $y'(0) = y'(2\pi)$ also.

Thus $\boxed{\lambda = 0 \text{ is an eigenvalue with } y_0 = 1 \text{ as an eigenfn.}}$

For $\lambda > 0$, let $\lambda = \mu^2$, $\mu \neq 0$

The general soln. is $y = c_1 \cos \mu x + c_2 \sin \mu x$

and $y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$

The boundary cond. $\Rightarrow \begin{cases} (1 - \cos 2\pi\mu) c_1 - (\sin 2\pi\mu) c_2 = 0 \\ (\sin 2\pi\mu) c_1 + (1 - \cos 2\pi\mu) c_2 = 0 \end{cases}$

For non-trivial soln., $\begin{vmatrix} 1 - \cos 2\pi\mu & -\sin 2\pi\mu \\ \sin 2\pi\mu & 1 - \cos 2\pi\mu \end{vmatrix} = 0$

i.e. $(1 - \cos 2\pi\mu)^2 + \sin^2 2\pi\mu = 0$

i.e. $\cos 2\pi\mu = 1$ i.e. $\mu = \pm n$, $n \in \mathbb{N}$.

$\boxed{\text{So, } \lambda_n = n^2 \text{ is an eigenvalue for each } n \in \mathbb{N} \text{ and } y_n = \cos nx \; ; \; \sin nx \text{ are eigenfns. for } n \in \mathbb{N}.}$