QWA

1. Discuss the existence and uniqueness of solutions and find the maximal interval of existence for the IVP: $ty' = t + |y|, y(-1) = 1$.

   Answer: This is not a linear or separable equation. The non-linearity $f(t, y) = \frac{t+|y|}{t}$ is locally Lipschitz in any closed rectangle that does not intersect the line $t = 0$. So By Picard's theorem the solution exists and unique in an interval around -1.

   To find the maximal interval, We notice that the nonlinearity is Lipschitz in domain $(-\infty, -\delta] \times I\!R$ for any $\delta > 0$. Indeed,

$$|f(t,x) - f(t,y)| = \frac{1}{|t|}||x| - |y|| \leq \frac{1}{\delta}|x - y|.$$

   Therefore, By global existence theorem, solution uniquely exits in $(-\infty, -\delta]$ for any $\delta > 0$. So the maximal interval of existence is the union of all these intervals which is $(-\infty, 0)$.

2. Discuss the existence and uniqueness of solutions and find the maximal interval of existence guaranteed by Picard's theorem for the IVP: $y' = t + y^2, y(0) = 0$.

   Answer: The nonlinearity $f(t, y) = t + y^2$ is locally Lipschitz in the rectangle

$$R_{ab} = \{(t, y) : |t| \leq a, |y| \leq b\}$$

   so by Picard's theorem, there exists solution in the interval $[-h, h]$ where $h = \min(a, b/M)$, $M = a + b^2$. So taking the function $h(x) = \frac{x}{(a+x^2)}$, we see that maximum of h is at $\frac{1}{2\sqrt{a}}$. Therefore, $h = \min\{a, \frac{1}{2\sqrt{a}}\}$. We have to find largest a such that we get h maximum. i.e., $h = \max_a \min\{a, \frac{1}{2\sqrt{a}}\} = \sqrt[3]{\frac{1}{4}}$.

3. Discuss the existence and uniqueness of the problem and then find the solution.

$$y' = 1 + y^2, \ y(0) = 1$$

   Answer: The function $f(t, y) = 1+y^2$ is locally Lipschitz in any bounded rectangle $R = \{(t, y) : |t| \leq a, |y - 1| \leq b\}$ around $(0, 1)$.

$$|\frac{\partial f}{\partial y}| \leq |2y| \leq 2(b+1).$$

Hence $f(t,x)$ is locally Liptschitz in R. So by Picard's theorem, there exists a unique solution in $[-h,h]$ where $h = \min(a, \frac{b}{M})$, $M = \max|f(t,x)|$. We can find the exact solution as the given equation is separable.

$$\frac{y'}{1+y^2} = 1$$

On integration, we get $\tan^{-1} y = t + c$ and $y = \tan(t+c)$ for $t + c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the one-parameter solution. substitute $t = 0$, we get $1 = \tan c$. So, the solution is $y = \tan(t + \frac{\pi}{4})$.

4. Solve the IVP
$$\frac{dy}{dt} = e^{(2t+y)}, \qquad y(0) = 0.$$
Answer: Note that $f(t,y)$ is continuous function and Lipschitz continuous w.r.t. y is any bounded interval containing $(0,0)$. So, we will have a unique solution in a neighborhood of $(0,0)$. By separating the variables, we get $e^{-y}dy = e^{2t}dt$. Integrating and simplifying results in $e^{-y} = -\frac{1}{2}e^{2t} - c_1$ for some constant c_1. Using $y(0) = 0$ we get $c_1 = -3/2$. Taking logarithm both sides we get,
$$y = -\ln\left(-\frac{1}{2}e^{2t} + 3/2\right)$$
Now domain of ln is $(0,\infty)$, so solution is valid in a domain where $e^{2t} < 3$ i.e. $t < \ln(3)/2 \approx 0.5493$.

5. The wood in a Egyptian burial case is found to contain 79% of carbon ^{14}C. What is the age of burial case. Half life of ^{14}C is 5730 years.
Answer: Decay of ^{14}C is modeled by $y' = ky$ and has solution, $y = y_0e^{kt}$ Now using $y(0) = 1$ and $y(5730) = 0.5$ we get,
$$k = \frac{\ln(0.5)}{5730} = -\frac{\ln(2)}{5730}.$$
Now solving $0.79 = e^{kt}$ for t we get $t = 1948.63$ years.