Tutorial 2 ::: MTL2005 (Algebra)

1. Let $\alpha \in \mathbb{Q}$. On $\mathbb{Q}$ define the binary operation $\star_\alpha$ as follows: $x \star_\alpha y = x + y + \alpha xy$. Find if
   (i) $\star_\alpha$ satisfies commutative law.
   (ii) $\star_\alpha$ satisfies associative law.
   (iii) there is an identity.
   (iv) $\mathbb{Q}$ is a group under $\star_\alpha$. If your answer is negative find the largest subset of $\mathbb{Q}$ which is a group under $\star_\alpha$.
2. Let $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{GL}(n, R) = \{A \in \mathrm{M}_n(R) : A \text{ is invertible}\}$ and let $\mathrm{SL}(n, R) = \{A \in \mathrm{M}_n(R) : \det A = 1\}$. Show that $\mathrm{GL}(n, R)$ is a group and $\mathrm{SL}(n, R)$ is a subgroup of $\mathrm{GL}(R, n)$.
3. Let $G = \left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0\right\}$. Show that G is a group under matrix multiplication.
4. Let G be a group. Show that the following statements are equivalent:
   (i) $(ab)^2 = a^2b^2$ for all $a, b \in G$.
   (ii) G is abelian.
   (iii) $(ab)^{-1} = a^{-1}b^{-1}$ for every $a, b \in G$.
5. Let X_n denote the first n natural numbers. Observe that every permutation on X_n is a permutation on X_m for $n \leq m$ that fixes every member of r larger than n. Use this idea to define an injective homomorphism from S_n to S_m for $n \leq m$.
6. Consider a regular n-gon Δ_n. Observe that there are exactly n rotations and n reflections that preserve Δ_n. Denote by x the anticlockwise rotation by angle $\frac{2\pi}{n}$ and denote by y, the reflection about a fixed line of symmetry. Show that any rotation is given by x^m and any reflection is $x^m y$ for $1 \leq m \leq n$.
7. Let $\psi : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{Z})$ be defined by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Is this a homomorphism? What is the kernel? Define a homomorphism from $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ to $\mathrm{GL}(2, \mathbb{Q})$ such that it is homomorphism from the multiplicative group $\mathbb{Q}^*$ to $\mathrm{GL}(2, \mathbb{Q})$.
8. Let $\psi : \mathbb{R} \to \mathrm{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ be defined by $\psi(t) = e^{2\pi i t}$. Find the kernel and the range.
9. Consider the map sgn: $S_n \to \{1, -1\}$ (called signature map) given by
$$\operatorname{sgn}(\sigma) = \frac{\prod_{1 \leq i < j \leq n}(x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}.$$
   Show that this is a group homomorphism with the obvious operations on S_n and $\{1, -1\}$. What is the kernel of this homomorphism?
10. Show that a nonempty subset X of G is a subgroup if and only if $x, y \in X \Rightarrow xy^{-1} \in X$. Use this criterion to show that $H, K < G \Rightarrow H \cap K < G$. Generalize to the following: If H_λ is a subgroup of G for every $\lambda \in J$, then the intersection $\cap_{\lambda \in J} H_\lambda$ is a subgroup of G. In particular, consider the family of subgroups $\Lambda_X := \{H < G : H \supset X\}$ for a subset $X \subset G$, use it to get the subgroup $\langle X \rangle := \cap_{\lambda \in \Lambda_X} H_\lambda$. Describe $\langle X \rangle$ (i) if $x = \{x\}$ (singleton) for any group, and (ii) if $X = \{(1\ 2\ 3\ 4), (1,\ 4)\}$ and $G = S_4$.
11. Let $H < G$. Define a relation $\sim$ on G by $x \sim y$ if $Hx = Hy$. Show that $\sim$ is an equivalence relation. Conclude that G is the union of (disjoint) cosets. Next, assume that $|G| < \infty$. Show that $|H|$ divides $|G|$ (this is the well-known Lagrange's Theorem) and the ratio $\frac{|G|}{|H|}$ is the number of distinct cosets of H in G.
12. Let $x \in G$. Define $o(x) = |\langle x \rangle|$ called the order. If $o(x) < \infty$, then show that $o(x)$ is the smallest natural number n with the property $x^n = 1$. Justify why $o(x)$ divides $o(G)$ if $o(G) < \infty$.
13. (I) Find the order of $e^{\frac{2\pi i k}{n}}$ in S^1 (see Q 9. for S^1).
    (II) Find the order of $(1\ 2\ 3\ \cdots m-1\quad m)$ in S_n for any $m \leq n$.
    (III) Show that $o(x) = o(x^{-1})$ and $o(x) = o(yxy^{-1})$ for every $x, y \in G$.

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(IV) If $o(x) = o(y)$, then is it true that $o(xy) = o(x)$?

(V) If $o(x), o(y) < \infty$, is it true that $o(xy) < \infty$? Consider both the cases, $xy = yx$ and $xy \neq yx$. (Find the order of $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$)

(VI) Let $x, y \in G$. If $o(x) = 4, o(y) = 2$ and $x^3 y = yx$, then find $o(xy)$.

(VII) Suppose $f: G \to H$ is a group homomorphism. Show that $o(x)$ is divisible by $o\big(f(x)\big)$.

(VIII) Consider the quotient group $G = \mathbb{Q}/\mathbb{Z}$. Find the order of $x + \mathbb{Z}$ for any $x \in \mathbb{Q}$.

(IX) Give example of an infinite group whose every element is of finite order.

(X) Suppose $x \in G$. If $x^6 = 1$ but $x^2 \neq 1$, then show that $x^4 \neq 1 \neq x^5$.

(XI) Suppose $x, \neq y \in G$ where G is an abelian group. If $o(x) = 2 = o(y)$, show that G has a subgroup of order 4.

14. Suppose the elements of the group G is listed as $1 = x_1, x_2, \ldots, x_n$. Then the table

	x_1	x_2	$\cdots x_j \cdots$	x_{n-1}	x_n
x_1					
x_2					
$\vdots$ x_i $\vdots$			$x_i x_j$		
x_{n-1}					
x_n					

Where the (i, j)-th entry is $x_i x_j$, is called the Cayley table of G. For $G = 1, a, b, c, d$ complete the following Cayley table.

	1	a	b	c	d
1	1				
a		b			e
b		c	d	e	
c		d		a	b
d					

15. Let G be a finite group. Show that the number of elements x of G such that $x^3 = 1$ is odd. Show that the number of elements x of G such that $x^2 \neq 1$ is even.
16. In a finite group, show that the number of non-identity elements that satisfy the equation $x^5 = 1$ is a multiple of 4.
17. Prove that if G is a group with the property that the square of every element is the identity, then G is abelian.
18. Let G be a group and let $g \in G$. Define a function ϕ_g from G to G by $\phi_g(x) = gxg^{-1}$ for all $x \in G$. Show that ϕ_g is an isomorphism (called an inner automorphism).
19. Let G be a finite group with more than one element. Show that G has an element of prime order.
20. Suppose $o(a) = 24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. More generally, find a generator of $\langle a^m \rangle \cap \langle a^n \rangle$.
21. Let $o(x) = 30$, $a = x^9$. Find the number of left cosets of $\langle a \rangle$ in $\langle x \rangle$. List them all.
22. Suppose that G is an abelian group of odd order. Show that the product of its elements is 1.
23. In a group of odd order, the equation $x^2 = a$ has a unique solution.
24. For a group G, let $Z(G) := \{x \in G : xy = yx \ \forall\, y \in G\}$. Show that $Z(G)$ is a normal subgroup of G.