Tutorial 3 ::: MTL2005 (Algebra)

1. Let p be a prime. Show that any group of order p is cyclic.
2. Suppose $H < G$ such that the index of H in G (the number of cosets of H in G) is 2. Show that $H \lhd G$.
3. Let C be a cyclic group of order n. Show the following statements.
   (a) If $H < C$, then H is cyclic.
   (b) If $d \in \mathbb{N}$ divides n, there is exactly one subgroup of C of order d.
   (c) The number of generators of a cyclic group of order d is $\phi(d)$, the number of natural numbers coprime to n that does not exceed n.
   (d) $\sum_{\substack{1 \le d < \le n \\ d|n}} \phi(d) = n$.
4. Show that $n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$ where $d = \gcd(m, n)$, and $m\mathbb{Z} \cap n\mathbb{Z} = l\mathbb{Z}$, where $l = \operatorname{lcm}(m, n)$. How many generators are there of the group $n\mathbb{Z}$?
5. Show that $S_n, (n \ge 3)$ is not cyclic. Show that
   (a) $S_3 = \langle (1 \ \ 2), (1 \ \ 2 \ \ 3) \rangle$.
   (b) $S_4 = \langle (1 \ \ 2), \ (1 \ \ 2 \ \ 3 \ \ 4) \rangle$
   (c) $S_n = \langle (1 \ \ 2), (1 \ \ 2 \ \ 3 \cdots \ \ n-1 \ \ n) \rangle$.
6. Show that $Z(G) \coloneqq \{x \in G : xy = yx \ \forall \ y \in G\}$ is a normal subgroup of G.
   (a) What is the centre of an abelian group?
   (b) Show that the centre of S_n is trivial for $n \ge 3$. What is $Z(S_2)$?
   (c) Show that the centre of A_n is trivial for $n \ge 4$. What is $Z(A_3)$?
   (d) Recall the Dihedral group of order $2n$ is given by $D_n = \langle x, y \mid x^2 = 1, xyx = y^{n-1} \rbrace$. Show that if $n \ge 3$ is odd then $Z(D_n)$ is trivial.
   (e) Show that $Z(D_n)$ has two elements if $n \ge 4$ is even. Find the elements explicitly.
   (f) Show that the centre of $SL(2, \mathbb{Z})$ consists of the identity element and it negative.
7. Let H<G. Define $N_G(H) = \{x \in G : xHx^{-1} = H\}$. Show the following statements.
   (a) $N_G(H) < G$.
   (b) $H \lhd N_G(H)$.
   (c) If K<G and $H \lhd K, then\ show\ that\ K < N_G(H)$.
   (d) $If\ G = S_3\ and\ H = \langle (1 \ \ 2) \rangle, Find\ N_G(H)$.
   (e) $If\ G = S_3\ and\ H = \langle (1 \ \ 2 \ \ 3) \rangle, Find\ N_G(H)$.
8. Let $G = \mathbb{Q}$.
   (a) Is G cyclic?
   (b) Show that G is not finitely generated. In other words, show that if $x_1, x_2, \dots, x_n \in \mathbb{Q}$, then $G \neq \langle x_1, x_2, \dots, x_n \rangle$.
   (c) If $x_1, x_2, \dots, x_n \in \mathbb{Q}$, then is there $y \in \mathbb{Q}$ such that $\langle x_1, x_2, \dots, x_n \rangle = \langle y \rangle$? (Hint. If $x_i = p_i/q_i$ where $q_1, q_2, \dots, q_n$ are pairwise coprime, then $\langle x_1, x_2, \dots, x_n \rangle < \left\langle \frac{1}{q_1 q_2 \cdots q_n} \right\rangle$).
   (d) Show that $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic groups.
9. Let G be a group. Define $G' = \langle xyx^{-1}y^{-1} : x, y \in G \rangle$, the smallest subgroup of G containing all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$. Show that
   (a) $G' \lhd G$.
   (b) G/G' is abelian.
   (c) If N is a normal subgroup of G such that G/N is abelian, then $G' \subset N$.
   (d) If K is an abelian group and $f: G \to K$ is a group homomorphism, then there is a {\bf unique} group homomorphism $\tilde{f} : \frac{G}{G'} \to K$, such that $\tilde{f} \circ \eta = f$, where η is the quotient map from G to G/G' so that $\eta(x) = G'x$.

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10. Let G be a group and let X be a non-empty set. If there is a group homomorphism f from G to the group of bijections of X (under composition of maps), then we say that G is acting on X via f.
    (a) Show that G **acts** on X if and only if there is a map from $G \times X$ to X (denote the image of $(g, x) \in G \times X$ in X by $g \star x$) such that
        1. $1 \star x = x$ and
        2. $g_1 g_2 \star x = g_1 \star (g_2 \star x)$.
    (b) Define, for $x \in X$, the **orbit** of x by $\mathcal{O}_x \coloneqq \{g \star x : g \in G\}$. Show that orbits partition X into disjoint parts (i.e., X is the union of orbits and any two orbits are identical of disjoint).
    (c) Define, for $x \in X$, the **stabilizer** of x by $Stb(x) \coloneqq \{g \in G : g \star x = x\}$. Show that $Stb(x) < G$.
    (d) The action of G on X is said to be **transitive** if given any elements $x, y \in X$, there exists $g \in X$, such that $y = g \star x$. Show that an action of a group on a nonempty set is transitive if and only if there is exactly one orbit.
    (e) Let $x \in X$. Define $\mu: G \to \mathcal{O}_x$ by $\mu(g) = g \star x$.
        1. If $y \in \mathcal{O}_x$ and $y = h \star x$ for $h \in G$, then show that $Stb(y) = h\, Stb(x)\, h^{-1}$.
        2. Show that $\nu: \frac{G}{Stb(x)} \to \mathcal{O}_x$ given by $\nu\big(gStb(x)\big) = g \star x$ is a bijective map.
        3. Show that if G is finite $|G|$ is divisible by $|\mathcal{O}_y|$ for every $y \in X$ and their ratio is the order of the stabilizer of y.
11. Let $G = S_n$ and let $X = \{1, 2, \ldots, n\}$. Consider the natural action of S_n on X. Find, for $k \in X$, the orbit $\mathcal{O}_k$ and the stabilizer $Stb(k)$.
12. Let $G = SL(2, \mathbb{Z})$ and $X = \mathbb{C}$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $z \in \mathbb{C} \setminus \{0\}$, then define $g \star z := \frac{az+b}{cz+d}$.
    (a) Show that $x \mapsto g \star x$ is an action of G on X.
    (b) Find the stabilizer group of i.
    (c) Is the action transitive?
    (d) Let $\mathcal{H} \coloneqq \{z \in \mathbb{C} : Im(z) > 0\}$ which is called the upper half plane. Show that $Im(g \star z) = \frac{Im(z)}{|cz+d|^2}$ and deduce that if $z \in \mathcal{H}$, then $g \star z \in \mathcal{H}$.

13. Let G be a group and X is the underlying set of G. For $g \in G$ and $x \in X$, define $g \star x = gxg^{-1}$. Show the following.
    (a) $Stb(x) = C_G(x)$, the centralizer of x in G (the collection of all elements of G that commute with x).
    (b) $\mathcal{O}_x = Cl(x)$, the conjugacy class containing x that consists of all conjugates of x (that is, the set $\{gxg^{-1} : g \in G\}$).
    (c) $\{x \in G : Cl(x) = \{x\}\} = Z(G)$, the centre of G (that is, the centre of group consists of precisely those elements of G which commute with every element of G).
    (d) $x \in Z(G) \Leftrightarrow |Cl(x)| = 1$.
    (e) Let $x_1, x_2, \ldots x_k \in G$ be such that $|Cl(x_i)| \neq 1$, $Cl(x_i) \cap Cl(x_j) = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{k} Cl(x_i) = G \setminus Z(G)$. Then (deduce the class equation):

$$|G| = |Z(G)| + \sum_{i=1}^{k} [G : C_G(x_i)]$$

14. Find all conjugacy classes of Q_8, the quaternion group and verify the class equation.

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15. Show that if $G/Z(G)$ is cyclic, then show that G is abelian. Use this to show the following: If G is a non-abelian group of order p^3 for some prime p, and $Z(G) \neq \{1\}$, then show that $|Z(G)| = p$.
16. If the order of G is pq, where p, q are both primes (possibly, equal), then show that $o(Z(G) = 1$ or pq.
17. Consider the map $l: G \to Bij(G)$ (the group of bijections of G onto itself under composition of maps), given by $g \mapsto l_g$ where $l_g(x) = gx$, the left multiplication by g. Show that
    (a) l_g is a bijection from G onto itself.
    (b) l is a homomorphism of groups.
    (c) l is injective.
    (d) Conclude that every group of order $n \in \mathbb{N}$ is isomorphic to a subgroup of S_n. (This is the well-known Cayley's Theorem).
    (e) What is the orbit and the stabilizer of an element of G?
18. Let G be a group having p^n elements where $p, n \in \mathbb{N}$, p is a prime. (We call a group having p^n elements a p-group). Show the following:
    (a) For any $g \in G$, either $|Cl(g)|$ is p^k for $0 \leq k < n$.
    (b) $|Z(G)| \geq p$.
    (c) If $n \leq 2$, then G is abelian. (Hint. Suppose $G \neq Z(G)$. Let $x \in G \backslash Z(G)$. Then $H = C_G(x)$, the centralizer of x contains $Z(G)$ properly and $H \neq G$.
19. Exhibit a Sylow 2-subgroup of S_4. Describe an isomorphism from this group to D_4
20. If $|G| = 36$, and G is non-abelian, prove that there is more than one Sylow 2-subgroup or there is more than one Sylow 3-subgroup.
21. How many abelian non-isomorphic groups are there of order 96? (Use structure theorem of finite abelian groups.)