

Tutorial 4 :: MTL2005/MTL105

1. Describe all ideals of $\mathbb{Z}$, $n\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{Q}[x]$, $\mathbb{Z}[i]$.
2. Give examples of rings where not every left ideal is a right ideal and not every right ideal is a left ideal.
3. Let I, J be left ideals of a ring R . Show that the following sets are left ideals of R .
 - (i) $I + J := \{x + y : x \in I, y \in J\}$.
 - (ii) $I \cap J$.
4. Let R be a ring.
 - a. Show that the intersection of any collection of (left or right or two-sided) ideals of a ring is (respectively, left or right or two-sided) an ideal of R .
 - b. What is the intersection of all (left, right or two-sided) ideals?
 - c. Let $a \in R$. Describe a typical element of the intersection of all (left or right or two-sided) ideals containing a .
5. An ideal in a ring is called a principal ideal if only one element generates the ideal. Find a necessary and sufficient condition for two ideals of $\mathbb{Z}$ which are related by containment relation (that is, if I, J are ideals of $\mathbb{Z}$, then give a condition for $I \subset J$).
6. Let S be a subset of a commutative ring R . Denote by $\langle S \rangle$ the intersection of all ideals of R that contain the given set S . If $S = \{a_1, a_2, \dots, a_n\}$, how does a typical element of R look? If R is a commutative ring with unity how does the form get modified?
7. Suppose $\psi: R \rightarrow S$ is a ring homomorphism.
 - (i) Show that $\ker \psi := \{x \in R : \psi(x) = 0\}$ (called the kernel of ψ) is an ideal of R .
 - (ii) Give an example to show that if I is an ideal of R , $\psi(I)$ is not necessarily an ideal of S . Find a condition under which $\psi(I)$ is an ideal of S .
 - (iii) Show that if J is an ideal of S , then $\psi^{-1}(J)$ is an ideal of R .
 - (iv) If $\mathcal{P}$ is a prime ideal of S , then show that $\psi^{-1}(\mathcal{P})$ is a prime ideal of R .
8. Suppose R is a commutative ring with unity. Show using the definition that every maximal ideal of R is a prime ideal of R .
9. Suppose I is an ideal of R . Show that $I + xI + y = I + xy$ defines a binary operation on the set of all cosets I in R . Hence deduce that R/I is a ring.
10. An element $u \in R$ is said to be a unit of the ring R (a ring with unity), if u is invertible with respect to multiplication, that is, there exists $v \in R$, such that $uv = vu = 1$. Show that the set $U(R)$ of all units forms a group with respect to multiplication. Find $U(\mathbb{Z}_n)$ for any $n = 2, 3, 4, \dots$
11. An element $x \in R \setminus \{0\}$ in a commutative ring R , is called a zero divisor if there exists $y \in R \setminus \{0\}$ such that $xy = 0$. (We can define what is a left zero divisor and a right zero divisor for a non-commutative ring.)

Remark: We shall not consider zero to be a zero divisor to avoid writing nonzero zero divisors for other zero divisors.
12. A commutative ring with unity is called an integral domain if it has no zero divisors. Find all zero divisors of $\mathbb{Z}_n$ for every $n \in \{2, 3, 4, \dots\}$
13. An integral domain is called a field if and only if every nonzero element is a unit. Under what condition on n , the ring of congruence classes modulo n is a field?
14. If $\mathbb{F}$ is a subring of the field $\mathbb{K}$, then $\mathbb{F}$ is called a subfield of $\mathbb{K}$ and $\mathbb{K}$ is called an extension of the field $\mathbb{F}$. Show that if $\mathbb{F}$ is a subfield of the ring R , then R is a vector space over $\mathbb{F}$. The vector space dimension of an extension R of $\mathbb{F}$ is called the extension degree.
15. Suppose $\mathbb{F}$ is a field.
 - (i) If $n1 = 0$ for a positive integer n , then show that the $p = \min\{n \in \mathbb{N} : n1 = 0 \text{ in } \mathbb{F}\}$ is a prime number.

Tutorial 4 ::: MTL2005/MTL105

- (ii) If p is a prime such that $p1 = 0$, then show that the map defined by $1 \mapsto 1$ from $\mathbb{Z}$ to $\mathbb{F}$ is a ring homomorphism with kernel $p\mathbb{Z}$. Use the first isomorphism theorem to show that $\mathbb{F}$ contains a copy of $\mathbb{Z}_p$.
 - (iii) If $n1 \neq 0$ for every $n \in \mathbb{N}$, then show that $\mathbb{F}$ contains a copy of $\mathbb{Q}$, the field of rational numbers.
16. Let R is a commutative ring with unity.
- (i) Show that I is a prime ideal if and only if the quotient R/I is an integral domain.
 - (ii) Show that J is a maximal ideal if and only if the quotient R/J is a field.
17. Let R be an integral domain. An element $x \in R \setminus (\{0\} \cup U(R))$ is called irreducible if $x = yz \Rightarrow y \in U(R)$ or $z \in U(R)$. An element $p \in R \setminus (\{0\} \cup U(R))$ is called a prime if $p|ab \Rightarrow p|a$ or $p|b$.
- (i) What are all irreducible elements of $\mathbb{Z}$?
 - (ii) What are all primes of $\mathbb{Z}$?
 - (iii) Show that in any integral domain every prime is irreducible.
 - (iv) Give an example to show that in some integral domains, not every irreducible element is a prime.
18. An integral domain R is called a Euclidean domain if there is a map (called Euclidean map) $d: R \setminus \{0\} \rightarrow \mathbb{N} \cup \{0\}$ with the following properties: I. $d(ab) \geq d(a)$ for any $a, b \in R \setminus \{0\}$ and II. given any $a \in R$, $b \in R \setminus \{0\}$, there exist $q, r \in R$ such that $a = bq + r$ where $r = 0$ or $d(r) < d(b)$.
- Show that
- (i) $\mathbb{Z}$ is a Euclidean domain with $d(a) = |a|$.
 - (ii) $\mathbb{Z}[i]$ is a Euclidean domain with $d(a + bi) = a^2 + b^2$.
 - (iii) For any field $\mathbb{F}$, the ring $\mathbb{F}[x]$ are Euclidean domain with $d(p(x)) = \deg p(x)$, the degree of the polynomial for any nonzero polynomial $p(x)$.
 - (iv) Show that any Euclidean domain is a principal ideal domain (that is, every ideal is generated by only one element).
19. Let R be a subring of S . Define $f: R[x] \rightarrow S$ by $f(p(x)) = p(\alpha)$ for a fixed element $\alpha \in S$.
- (i) Show that f is a ring homomorphism.
 - (ii) $\alpha \in R$ if and only if the Range of f is R .
 - (iii) Set $R = \mathbb{Q}, S = \mathbb{C}$. Define $\mathbb{Q}[\alpha]$ to be the range of f . Show that
 - (a) $\mathbb{Q}[\alpha]$ is a field if and only if α is a root of a polynomial with coefficients from $\mathbb{Q}$. In this case describe how an element of $\mathbb{Q}[\alpha]$ can be **uniquely** written.
 - (b) If $\mathbb{Q}[\alpha]$ is an infinite dimensional vector space over $\mathbb{Q}$ if $\mathbb{Q}[\alpha]$ is not a field. In this case describe how an element of $\mathbb{Q}[\alpha]$ can be **uniquely** written.
 - (iv) Set $R = \mathbb{Z}, S = \mathbb{C}$.
 - (a) Describe all elements of $\mathbb{Z}\left[\frac{1}{n}\right]$ for $n \in \mathbb{N}$.
 - (b) Describe all elements of $\mathbb{Z}[\sqrt{-5}]$.
 - (c) Describe all elements of $\mathbb{Z}[\alpha]$ where α is a cube root of 2.
20. **In the family of principal ideals** of an integral domain R , show that the ideal generated by an irreducible element is a maximal ideal.
21. If R, S are rings, then $R \times S$ is a ring with component wise addition and multiplication.
- (i) If both R and S are rings with unity, then show that $R \times S$ is a ring with unity.
 - (ii) If $|R|, |S| > 1$, can $R \times S$ be an integral domain?
 - (iii) Describe $U(R \times S)$ with the help of $U(R)$ and $U(S)$.
 - (iv) What are all zero divisors of $\mathbb{F} \times \mathbb{K}$ if $\mathbb{F}$ and $\mathbb{K}$ are both fields?
 - (v) What are all zero divisors of $R \times S$?