ON UNIT GROUP OF FINITE SEMISIMPLE GROUP ALGEBRAS OF NON-METABELIAN GROUPS OF ORDER 108

Gaurav Mittal$^1$, R. K. Sharma$^2$

$^1$Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India
$^2$Department of Mathematics, Indian Institute of Technology Delhi, New Delhi, India
email: $^1$gmittal@ma.iitr.ac.in, $^2$rksharma@maths.iitd.ac.in

Abstract. In this paper, we characterize the unit groups of semisimple group algebras $\mathbb{F}_qG$ of non-metabelian groups of order 108, where $\mathbb{F}_q$ is a field with $q = p^k$ elements for some prime $p > 3$ and positive integer $k$. Upto isomorphism, there are 45 groups of order 108 but only 4 of them are non-metabelian. We consider all the non-metabelian groups of order 108 and find the Wedderburn decomposition of their semisimple group algebras. And as a by-product obtain the unit groups.

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1. Introduction

Let $\mathbb{F}_q$ denote a finite field with $q = p^k$ elements for odd prime $p > 3$, $G$ be a finite group and $\mathbb{F}_qG$ be the group algebra. The study of the unit groups of group algebras is a classical problem and has applications in cryptography [4] as well as in coding theory [5] etc. For the exploration of Lie properties of group algebras and isomorphism problems, units are very useful see, e.g. [1]. We refer to [11] for elementary definitions and results about the group algebras and [2, 15] for the abelian group algebras and their units. Recall that a group $G$ is metabelian if there is a normal subgroup $N$ of $G$ such that both $N$ and $G/N$ are abelian. The unit groups of the finite semisimple group algebras of metabelian groups have been well studied.

In this paper, we are concerned about the unit groups of the group algebras of non-metabelian groups. Let us first mention the available literature in this direction. From [14], we know that all the groups up to order 23 are metabelian. The only non-metabelian groups of order 24 are $S_4$ and $SL(2, 3)$ and the unit group of their group algebras have been discussed in [7, 9]. Further, from [14], we also deduce that there are non-metabelian groups of order of 48, 54, 60, 72, 108 etc. The unit groups of the group algebras of non-metabelian groups up to order 72 have been discussed in [12, 10]. The unit group of the semisimple group algebra of the non-metabelian group $SL(2, 5)$ has been discussed in [13].

The main motive of this paper is to characterize the unit groups of $\mathbb{F}_qG$, where $G$ represents a non-metabelian group of order 108. It can be verified that, upto isomorphism, there are 45 groups of order 108 and only 4 of them are non-metabelian. We deduce the Wedderburn
decomposition of group algebras of all the 4 non-metabelian groups and then characterize the respective unit groups. The rest of the paper is organized in following manner: We recall all the basic definitions and results to be needed in our work in Section 2. Our main results on the characterization of the unit groups are presented in third section and the last section includes some discussion.

2. Preliminaries

Let \( e \) denote the exponent of \( G \), \( \zeta \) a primitive \( e^{th} \) root of unity. On the lines of [3], we define

\[ I_\mathbb{F} = \{ n \mid \zeta \mapsto \zeta^n \text{ is an automorphism of } \mathbb{F}(\zeta) \text{ over } \mathbb{F} \}, \]

where \( \mathbb{F} \) is an arbitrary finite field. Since, the Galois group \( \text{Gal}(\mathbb{F}(\zeta), \mathbb{F}) \) is a cyclic group, for any \( \tau \in \text{Gal}(\mathbb{F}(\zeta), \mathbb{F}) \), there exists a positive integer \( s \) which is invertible modulo \( e \) such that \( \tau(\zeta) = \zeta^s \). In other words, \( I_\mathbb{F} \) is a subgroup of the multiplicative group \( \mathbb{Z}_e^* \) (group of integers which are invertible with respect to multiplication modulo \( e \)). For any \( p \)-regular element \( g \in G \), i.e. an element whose order is not divisible by \( p \), let the sum of all conjugates of \( g \) be denoted by \( \gamma_g \), and the cyclotomic \( \mathbb{F} \)-class of \( \gamma_g \) be denoted by \( S(\gamma_g) = \{ \gamma_g^n \mid n \in I_\mathbb{F} \} \). The cardinality of \( S(\gamma_g) \) and the number of cyclotomic \( \mathbb{F} \)-classes will be incorporated later on for the characterization of the unit groups.

Next, we recall two important results from [3]. First one relates the number of cyclotomic \( \mathbb{F} \)-classes with the number of simple components of \( \mathbb{F}G/J(\mathbb{F}G) \). Here \( J(\mathbb{F}G) \) denotes the Jacobson radical of \( \mathbb{F}G \). Second one is about the cardinality of any cyclotomic \( \mathbb{F} \)-class in \( G \).

**Theorem 2.1.** The number of simple components of \( \mathbb{F}G/J(\mathbb{F}G) \) and the number of cyclotomic \( \mathbb{F} \)-classes in \( G \) are equal.

**Theorem 2.2.** Let \( \zeta \) be defined as above and \( j \) be the number of cyclotomic \( \mathbb{F} \)-classes in \( G \). If \( K_i, 1 \leq i \leq j \), are the simple components of center of \( \mathbb{F}G/J(\mathbb{F}G) \) and \( S_i, 1 \leq i \leq j \), are the cyclotomic \( \mathbb{F} \)-classes in \( G \), then \( |S_i| = [K_i : \mathbb{F}] \) for each \( i \) after suitable ordering of the indices if required.

To determine the structure of unit group \( U(\mathbb{F}G) \), we need to determine the Wedderburn decomposition of the group algebra \( \mathbb{F}G \). In other words, we want to determine the simple components of \( \mathbb{F}G \). Based on the existing literature, we can always claim that \( \mathbb{F} \) is one of the simple component of decomposition of \( \mathbb{F}G/J(\mathbb{F}G) \). The simple proof is given here for the completeness.

**Lemma 2.1.** Let \( A_1 \) and \( A_2 \) denote the finite dimensional algebras over \( \mathbb{F} \). Further, let \( A_2 \) be semisimple and \( g \) be an onto homomorphism between \( A_1 \) and \( A_2 \), then we must have \( A_1/J(A_1) \cong A_3 + A_2 \), where \( A_3 \) is some semisimple \( \mathbb{F} \)-algebra.
Proof. From [6], we have $J(A_1) \subseteq \text{Ker}(g)$. This means there exists $F$-algebra homomorphism $g_1$ from $A_1/J(A_1)$ to $A_2$ which is also onto. In other words, we have
\[
g_1 : A_1/J(A_1) \rightarrow A_2 \text{ defined by } g_1(a + J(A_1)) = g(a), \quad a \in A_1.
\]
As $A_1/J(A_1)$ is semisimple, there exists an ideal $I$ of $A_1/J(A_1)$ such that
\[
A_1/J(A_1) = \text{Ker}(g_1) \oplus I.
\]
Our claim is that $I \cong A_2$. For this to prove, note that any element $a \in A_1/J(A_1)$ can be uniquely written as $a = a_1 + a_2$ where $a_1 \in \text{Ker}(g_1), a_2 \in I$. So, define
\[
g_2 : A_1/J(A_1) \hookrightarrow \text{Ker}(g_1) \oplus A_2 \text{ by } g_2(a) = (a_1, g_1(a_2)).
\]
Since, $\text{Ker}(g_1)$ is a semisimple algebra over $F$ and $A_2$ is an isomorphic $F$-algebra, claim and the result holds.

Above lemma concludes that $F$ is one of the simple components of $FG$ provided $J(FG) = 0$. Now, we recall a result which characterize the set $I_F$ defined in the beginning of this section. For proof, see Theorem 2.21 in [8].

**Theorem 2.3.** Let $F$ be a finite field with prime power order $q$. If $e$ is such that $\gcd(e, q) = 1$, $\zeta$ is the primitive $e^{th}$ root of unity and $|q|$ is the order of $q$ modulo $e$, then modulo $e$, we have $I_F = \{1, q, q^2, \ldots, q^{|q| - 1}\}$.

Next result is Proposition 3.6.11 from [11] and is useful for the determination of commutative simple components of the group algebra $F_qG$.

**Theorem 2.4.** If $RG$ is a semisimple group algebra, then $RG \cong R(G/G') \oplus \Delta(G, G')$, where $G'$ is the commutator subgroup of $G$, $R(G/G')$ is the sum of all commutative simple components of $RG$, and $\Delta(G, G')$ is the sum of all others.

We end this section by recalling a Proposition 3.6.7 from [11] which is a generalized version of the last result.

**Theorem 2.5.** Let $RG$ be a semisimple group algebra and $H$ be a normal subgroup of $G$. Then $RG \cong R(G/H) \oplus \Delta(G, H)$, where $\Delta(G, H)$ is a left ideal of $RG$ generated by the set $\{h - 1 : h \in H\}$.

3. Unit group of $F_qG$ where $G$ is a non-metabelian group of order 108

The main objective of this section is to characterize the unit group of $F_qG$ where $G$ is a non-metabelian group of order 108. Upto isomorphism, there are 4 non-metabelian groups of order 108 namely: $(1)\ G_1 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4. \quad (2)\ G_2 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4.\n\quad (3)\ G_3 = ((C_3 \times C_3) \rtimes C_3) \rtimes (C_2 \times C_2). \quad (4)\ G_4 = C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2).$
Here $G_1$ and $G_2$ are two non-isomorphic groups formed by the semi-direct product of $(C_3 \times C_3) \rtimes C_3$ and $C_4$ which will be clear once we discuss the presentation of these groups. We consider each of these 4 groups one by one and discuss the unit groups of their respective group algebras along with the Wedderburn decompositions in subsequent subsections. Throughout this paper, we use the notation $[x, y] = x^{-1}y^{-1}xy$.

### 3.1. The group $G_1 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4$.

Group $G_1$ has the following presentation:

$$G_1 = \langle x, y, z, t \mid x^2y^{-1}, [y, x], [z, x]z^{-1}, [w, x]x^{-1}, [t, x], y^2, [z, y], [w, y], [t, y], z^3, [w, z]z^{-1}, [t, z], w^3, [t, w], t^3 \rangle.$$

Also $G_1$ has 20 conjugacy classes as shown in the table below.

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where r, s and o denote the representative of conjugacy class, size and order of the representative of the conjugacy class, respectively. From the above discussion, it is clear that exponent of $G_1$ is 12. Also $G_1' \cong (C_3 \times C_3) \rtimes C_3$ with $G_1/G_1' \cong C_4$. Next, we discuss the unit group of the group algebra $\mathbb{F}_qG_1$ when $p > 3$.

**Theorem 3.1.** The unit group of $\mathbb{F}_qG_1$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:

1. For any $p$ and $k$ even or $p^k \equiv 1 \mod 12$ with $k$ odd, we have

   $$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^8.$$

2. For $p^k \equiv 5 \mod 12$ with $k$ odd, we have

   $$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q^2)^4.$$

3. For $p^k \equiv 7 \mod 12$ with $k$ odd, we have

   $$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^2 \oplus F_{q^2}^* \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q^4)^4 \oplus GL_3(\mathbb{F}_q^2)^2.$$

4. For $p^k \equiv 11 \mod 12$ with $k$ odd, we have

   $$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^2 \oplus F_{q^2}^* \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q^2)^4.$$

**Proof.** Since $\mathbb{F}_qG_1$ is semisimple, using Lemma 2.1 we get

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r), \text{ for some } t \in \mathbb{Z}. \quad (3.1)$$

First assume that $k$ is even which means for any prime $p > 3$, we have

$$p^k \equiv 1 \mod 3 \text{ and } p^k \equiv 1 \mod 4.$$
Using Chinese remainder theorem, we get \( p^k \equiv 1 \mod 12 \). This means \( |S(\gamma_g)| = 1 \) for each \( g \in G_1 \) as \( I_F = \{1\} \). Hence, equation (3.1), Theorems 2.1 and 2.2 imply that

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{19} M_{n_r}(\mathbb{F}_q). \tag{3.2}
\]

Incorporating Theorem 2.4 with \( G'_1 \cong (C_3 \times C_3) \rtimes C_3 \) in equation (3.2) to obtain

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus_{r=1}^{16} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{16} n_r^2. \tag{3.3}
\]

Above equation gives the only possibility \((2^8, 3^8)\) for the values of \( n_r \)'s where \( a^b \) means \((a, a, \cdots, b \text{ times})\) and therefore, equation (3.3) implies that

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^8. \tag{3.4}
\]

Now we consider that \( k \) is odd. We shall discuss this possibility in the following four cases:

Case 1. \( p^k \equiv 1 \mod 3 \) and \( p^k \equiv 1 \mod 4 \). In this case, Wedderburn decomposition is again given by equation (3.4).

Case 2. \( p^k \equiv 5 \mod 12 \). In this case, we have

\[ S(\gamma_t) = \{\gamma_t, \gamma_t^2\}, S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xt^2}\}, S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yt^2}\}, S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xyt^2}\}, \]

and \( S(\gamma_g) = \{\gamma_g\} \) for the remaining representatives \( g \) of conjugacy classes. Therefore, equation (3.1) and Theorems 2.1, 2.2 imply that

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{t=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{t=9}^{15} M_{n_r}(\mathbb{F}_q^2). \]

Since \( G_1/G'_1 \cong C_4 \), we have \( \mathbb{F}_q C_4 \cong \mathbb{F}_q^4 \). This with above and Theorem 2.4 yields

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r}(\mathbb{F}_q^2), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{8} n_r^2 + 2 \sum_{r=9}^{12} n_r^2. \tag{3.5}
\]

Further, consider the normal subgroup \( H_1 = \langle t \rangle \) of \( G_1 \) having order 3 with \( K_1 = G_1/H_1 \cong (C_3 \times C_3) \rtimes C_4 \). The quotient group \( K_1 \) has 12 conjugacy classes as shown in the table below.

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It can be verified that for all the representatives \( g \) of \( K_1, |S(\gamma_g)| = 1 \). Therefore, from Theorems 2.1 and 2.2, we have

\[
\mathbb{F}_q K_1 \cong \mathbb{F}_q \oplus_{t=1}^{11} M_{t_r}(\mathbb{F}_q), \ t_r \in \mathbb{Z}. \]

Observe that \( K_1/K'_1 \cong C_4 \). So, Theorem 2.4 implies that

\[
\mathbb{F}_q K_1 \cong \mathbb{F}_q^4 \oplus_{t=1}^{8} M_{t_r}(\mathbb{F}_q), \text{ with } 32 = \sum_{r=1}^{8} t_r^2, t_r \geq 2.
\]
This gives us the only choice (2^8) for values of \( t_i' \)'s and therefore, Theorem 2.5 and equation (3.5) yields

\[ F_q G_1 \cong F_q^4 \oplus M_2(F_q)^8 \oplus \bigoplus_{r=1}^4 M_n, (F_q^2), \quad n_r \geq 2 \text{ with } 36 = \sum_{r=1}^4 n_r^2. \]

Above leaves us with the only choice (3^4) for values of \( n_i' \)'s which means the required Wedderburn decomposition is

\[ F_q G_1 \cong F_q^4 \oplus M_2(F_q)^8 \oplus M_3(F_q^2)^4. \]

Case 3. \( p^k \equiv 7 \mod 12 \). In this case, we have

\[ S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}, \quad S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}, \quad S(\gamma_{xt^2}) = \{\gamma_{xt^2}, \gamma_{xyt^2}\}, \quad \text{and } S(\gamma_y) = \{\gamma_y\} \]

for the remaining representatives \( g \) of conjugacy classes. Therefore, equation (3.1) and Theorems 2.1, 2.2 imply that

\[ F_q G_1 \cong F_q \oplus F_q^1 \oplus F_q^2 \oplus F_q^4 \oplus F_q^6 \oplus F_q^8 \oplus F_q^{10} \oplus M_{n_r}(F_q). \]

Since \( G_1/G'_1 \cong C_4 \), we have \( F_q C_4 \cong F_q^2 \oplus F_q^4 \). This with above and Theorem 2.4 yields

\[ F_q G_1 \cong F_q^2 \oplus F_q^4 \oplus F_q^6 \oplus F_q^8 \oplus F_q^{10} \oplus M_{n_r}(F_q) \oplus M_{n_r}(F_q^2), \quad n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{12} n_r^2 + 2 \sum_{r=13}^{14} n_r^2. \quad (3.6) \]

Again consider the normal subgroup \( H_1 \) of \( G_1 \). In this case, it can be verified that \( |S(\gamma_y)| = 1 \) for all the representatives \( g \) of \( K_1 \) except \( x \) and \( xy \) for which \( S(\gamma_x) = \{\gamma_x, \gamma_{xy}\} \). Therefore, employing Theorems 2.1 and 2.2 to obtain

\[ F_q K_1 \cong F_q \oplus F_q^9 \oplus F_q^{10} \oplus M_{t_r}(F_q) \oplus M_{t_{10}}(F_q^2), \quad t_r \in \mathbb{Z}. \]

Since \( K_1/K'_1 \cong C_4 \), above and Theorem 2.4 imply that

\[ F_q K_1 \cong F_q^2 \oplus F_q^4 \oplus F_q^6 \oplus \bigoplus_{r=1}^8 M_{t_r}(F_q), \quad \text{with } 32 = \sum_{r=1}^8 t_r^2, \quad t_r \geq 2. \]

This gives us the only choice (2^8) for values of \( t_i' \)'s. Hence, Theorem 2.5 and equation (3.6) imply that

\[ F_q G_1 \cong F_q^2 \oplus F_q^4 \oplus F_q^6 \oplus \bigoplus_{r=1}^8 M_{n_r}(F_q) \oplus M_{n_r}(F_q^2), \quad \text{with } 72 = \sum_{r=1}^4 n_r^2 + 2 \sum_{r=5}^6 n_r^2. \]

Above leaves us with the only choice (3^6) for values of \( n_i' \)'s which means the required Wedderburn decomposition is

\[ F_q G_1 \cong F_q^2 \oplus F_q^4 \oplus M_2(F_q)^8 \oplus M_3(F_q)^4 \oplus M_3(F_q^2)^2. \]

Case 4. \( p^k \equiv 11 \mod 12 \). In this case, we have

\[ S(\gamma_{i}) = \{\gamma_{i}, \gamma_{i^2}\}, \quad S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\}, \quad S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yt^2}\}, \quad S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xt^2}\}, \quad S(\gamma_{x}) = \{\gamma_{x}, \gamma_{xy}\}, \quad \text{and } S(\gamma_{y}) = \{\gamma_{y}\} \]
for the remaining representatives \( g \) of conjugacy classes. Therefore, equation (3.1) and Theorems 2.1, 2.2 imply that

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^8 M_{n_r} (\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r} (\mathbb{F}_q^2).
\]

Since \( G_1/G'_1 \cong C_4 \), we have \( \mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus M_2 (\mathbb{F}_q^2) \). This with above and Theorem 2.4 yields

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus_{r=1}^8 M_{n_r} (\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r} (\mathbb{F}_q^2), \quad n_r \geq 2 \text{ with } 104 = \sum_{r=1}^8 n_r^2 + 2 \sum_{r=9}^{12} n_r^2. \quad (3.7)
\]

In this case, again we have \( |S(\gamma_g)| = 1 \) for all representatives \( g \) of \( K_1 \) except \( x \) and \( xy \) which means the Wedderburn decomposition of \( \mathbb{F}_q K_1 \) is same as obtained in case 3, i.e.

\[
\mathbb{F}_q K_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus M_2 (\mathbb{F}_q)^8.
\]

Now employ Theorem 2.5 and equation (3.7) to obtain

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus M_2 (\mathbb{F}_q)^8 \oplus \mathbb{F}_q^2, \text{ with } 36 = \sum_{r=1}^4 n_r^2.
\]

This leaves us with the only choice \((3^4)\) for values of \( n_r \)s which means the required Wedderburn decomposition is

\[
\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus M_2 (\mathbb{F}_q)^8 \oplus M_3 (\mathbb{F}_q^2)^4.
\]

\[\square\]

3.2. The group \( G_2 = ((C_3 \times C_3) \times C_3) \times C_4 \). Group \( G_2 \) has the following presentation:

\[
G_2 = \langle x, y, z, w, t \mid x^2 y^{-1}, [y, x], [z, x]w^{-2}, [w, x]t^{-1}w^{-1}z^{-2}, [t, x], y^2, [z, y]z^{-1}, [w, y]t^{-1}w^{-1}, [t, y], z^3, [w, z]t^{-1}, [t, z], w^3, [t, w], t^3 \rangle.
\]

Further, \( G_2 \) has 14 conjugacy classes as shown in the table below.

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<th>t²</th>
<th>xyz</th>
<th>xyzw</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>o</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>3</td>
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<td>12</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

From above discussion, clearly the exponent of \( G_2 \) is 12. Also \( G'_2 \cong (C_3 \times C_3) \times C_3 \) and \( G_2/G'_2 \cong C_4 \). Next, we discuss the unit group of \( \mathbb{F}_q G_2 \) when \( p > 3 \).

Theorem 3.2. The unit group of \( \mathbb{F}_q G_2 \), for \( q = p^k \), \( p > 3 \) where \( \mathbb{F}_q \) is a finite field having \( q = p^k \) elements is as follows:

1. for any \( p \) and \( k \) even or \( p^k \equiv 1 \mod 12 \) with \( k \) odd, we have

\[
U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^4 \oplus GL_3 (\mathbb{F}_q)^8 \oplus GL_4 (\mathbb{F}_q^2)^2.
\]

2. for \( p^k \equiv 5 \mod 12 \) with \( k \) odd, we have

\[
U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^4 \oplus GL_4 (\mathbb{F}_q)^2 \oplus GL_3 (\mathbb{F}_q^2)^4.
\]
(3) for \(p^k \equiv 7 \mod 12\) with \(k\) odd, we have
\[ U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_q^2 \oplus GL_3(\mathbb{F}_q)^4 \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q^2)^2. \]

(4) for \(p^k \equiv 11 \mod 12\) with \(k\) odd, we have
\[ U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_q^2 \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q^2)^2. \]

Proof. Since \(\mathbb{F}_q G_2\) is semisimple, we have
\[ \mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_q), \text{ for some } t \in \mathbb{Z}. \] (3.8)

First assume that \(k\) is even which means for any prime \(p > 3\), \(p^k \equiv 1 \mod 12\). This means \(|S(\gamma_g)| = 1\) for each \(g \in G_2\). Hence, equation (3.8), Theorems 2.1 and 2.2 imply that
\[ \mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q). \]

Using Theorem 2.4 with \(G'_2 \cong (C_3 \times C_3) \times C_3\) to obtain
\[ \mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{10} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{10} n_r^2. \] (3.9)

Above equation gives us four possibilities \((2^8, 6^2), (2^5, 3^2, 4, 5^2), (2^4, 3^4, 4, 6)\) and \((3^8, 4^2)\) for the values of \(n_r\)’s. Further, consider the normal subgroup \(H_2 = \langle t \rangle\) of \(G_2\) having order 3 with \(K_2 = G_2/H_2 \cong (C_3 \times C_3) \times C_4\). It can be verified that \(K_2\) has 6 conjugacy classes as shown in the table below.

<table>
<thead>
<tr>
<th>(r)</th>
<th>(e)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(w)</th>
<th>(xy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>4</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>(o)</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Also, for all the representatives \(g\) of \(K_2\), \(|S(\gamma_g)| = 1\) which means by Theorems 2.1 and 2.2, we have
\[ \mathbb{F}_q K_2 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{5} M_{t_r}(\mathbb{F}_q), \text{ for } t_r \in \mathbb{Z}. \]

Observe that \(K_2/K'_2 \cong C_4\). This with above and Theorem 2.4 imply that
\[ \mathbb{F}_q K_2 \cong \mathbb{F}_q \oplus \bigoplus_{r=1}^{2} M_{t_r}(\mathbb{F}_q), \text{ with } 32 = \sum_{r=1}^{2} t_r^2, \text{ for } t_r \geq 2. \]

This gives us the only choice \((4^2)\) for values of \(t_r\). Therefore, Theorem 2.5 and equation (3.9) imply that \((3^8, 4^2)\) is the correct choice for values of \(n_r\)’s and therefore, we have
\[ \mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus M_3(\mathbb{F}_q)^8 \oplus M_4(\mathbb{F}_q)^2. \] (3.10)

Now we consider that \(k\) is odd. We shall discuss this possibility in the following four cases:

Case 1: \(p^k \equiv 1 \mod 12\). In this case, Wedderburn decomposition is given by equation (3.10).

Case 2. \(p^k \equiv 5 \mod 12\). In this case, we have
\[ S(\gamma_1) = \{\gamma_t, \gamma^2\}, S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{2t}\}, S(\gamma_{gw}) = \{\gamma_{gw}, \gamma_{yt}\}, S(\gamma_{xyz}) = \{\gamma_{xyz}, \gamma_{xyzw}\}, \]
and \( S(\gamma_g) = \{\gamma_g\} \) for the remaining representatives \( g \) of conjugacy classes. Therefore, equation (3.8) and Theorems 2.1, 2.2 imply that

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus_{r=1}^5 M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^9 M_{n_r}(\mathbb{F}_q^2).
\]

Since \( G_2/G_2' \cong C_4 \), we have \( \mathbb{F}_q C_4 \cong \mathbb{F}_q^4 \). This with above and Theorem 2.4 yields

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus_{r=1}^2 M_{n_r}(\mathbb{F}_q) \oplus_{r=3}^6 M_{n_r}(\mathbb{F}_q^2), \quad n_r \geq 2 \text{ with } 104 = \sum_{r=1}^2 n_r^2 + 2 \sum_{r=3}^6 n_r^2. \tag{3.11}
\]

Further, again consider the normal subgroup \( H_2 = \langle t \rangle \) of \( G_2 \). It can be verified that for all the representatives \( g \) of \( K_2 \), \( |S(\gamma_g)| = 1 \) which means (as earlier)

\[
\mathbb{F}_q K_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2.
\]

This with Theorem 2.5 and equation (3.11) imply that

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2 \oplus_{r=1}^4 M_{n_r}(\mathbb{F}_q^2), \quad n_r \geq 2 \text{ with } 36 = \sum_{r=1}^4 n_r^2.
\]

Above leaves us with the only choice \((3^4)\) for values of \( n_r \)'s which means the required Wedderburn decomposition is

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4.
\]

Case 3. \( p^k \equiv 7 \mod 12 \). In this case, we have

\[
S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}, \quad S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyzw}\}, \quad S(\gamma_xt) = \{\gamma_xt, \gamma_{xyz}\}, \quad \text{and } S(\gamma_g) = \{\gamma_g\}
\]

for the remaining representatives \( g \) of conjugacy classes. Therefore, equation (3.8) and Theorems 2.1, 2.2 imply that

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus_{r=1}^7 M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{10} M_{n_r}(\mathbb{F}_q^2).
\]

Since \( G_2/G_2' \cong C_4 \), we have \( \mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^3 \). This with Theorem 2.4 yields

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus_{r=1}^6 M_{n_r}(\mathbb{F}_q) \oplus_{r=7}^8 M_{n_r}(\mathbb{F}_q^2), \quad n_r \geq 2 \text{ with } 104 = \sum_{r=1}^6 n_r^2 + 2 \sum_{r=7}^8 n_r^2. \tag{3.12}
\]

Further, it can be verified that \( |S(\gamma_g)| = 1 \) for all the representatives \( g \) of \( K_2 \) except \( x \) and \( xy \). For these, we have \( S(\gamma_x) = \{\gamma_x, \gamma_{xy}\} \) which means by Theorems 2.1 and 2.2, we have

\[
\mathbb{F}_q K_2 \cong \mathbb{F}_q^3 \oplus M_{t_r}(\mathbb{F}_q) \oplus M_{t_r}(\mathbb{F}_q^2), \quad t_r \in \mathbb{Z}.
\]

Incorporating \( K_2/K_2' \cong C_4 \) with Theorem 2.4 to obtain

\[
\mathbb{F}_q K_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus_{r=1}^2 M_{t_r}(\mathbb{F}_q), \quad \text{with } 32 = \sum_{r=1}^2 t_r^2, \quad t_r \geq 2.
\]

This gives us only choice \((4^2)\) for values of \( t_r \)'s. Therefore, Theorem 2.5 and equation (3.12) yields

\[
\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_q^2 \oplus M_4(\mathbb{F}_q)^2 \oplus_{r=1}^4 M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^6 M_{n_r}(\mathbb{F}_q^2), \quad \text{with } 72 = \sum_{r=1}^4 n_r^2 + 2 \sum_{r=5}^6 n_r^2.
\]
Above leaves us with the only choice \((3^6)\) for values of \(n_r^\prime\) which means the required Wedderburn decomposition is
\[
F_qG_2 \cong F_q^2 \oplus F_q^2 \oplus M_4(F_q)^2 \oplus M_3(F_q)^4 \oplus M_3(F_q^2)^2.
\]

Case 4. \(p^k \equiv 11 \mod 12\). In this case, we have
\[
S(\gamma_t) = \{\gamma_t, \gamma_t^2\}, \quad S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}, \quad S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyz}\}, \quad S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyzw}\},
\]
for the remaining representatives \(g\) of conjugacy classes. Therefore, equation (3.8) and Theorems 2.1, 2.2 imply that
\[
F_qG_2 \cong F_q^2 \oplus_{r=1}^3 M_{n_r}(F_q) \oplus_{r=4}^8 M_{n_r}(F_q^2).
\]

Since \(G_2/G_2^\prime \cong C_4\), we have \(F_qC_4 \cong F_q^2 \oplus F_q^2\). This with above and Theorem 2.4 yields
\[
F_qG_2 \cong F_q^2 \oplus F_q^2 \oplus_{r=1}^2 M_{n_r}(F_q) \oplus_{r=3}^6 M_{n_r}(F_q^2), \quad n_r \geq 2 \text{ with } 104 = \sum_{r=1}^2 n_r^2 + 2 \sum_{r=3}^6 n_r^2. \tag{3.13}
\]
Further, it can be verified that \(|S(\gamma_g)| = 1\) for all the representatives \(g\) of \(K_2\) except \(x\) and \(xy\) which means (as in case 3),
\[
F_qK_2 \cong F_q^2 \oplus F_q^2 \oplus M_4(F_q)^2.
\]

Now employ Theorem 2.5 and equation (3.13) to obtain
\[
F_qG_2 \cong F_q^2 \oplus F_q^2 \oplus M_4(F_q)^2 \oplus_{r=1}^4 M_{n_r}(F_q^2), \text{ with } 36 = \sum_{r=1}^4 n_r^2.
\]

Above leaves us with the only choice \((3^4)\) for values of \(n_r^\prime\) which means the required Wedderburn decomposition is
\[
F_qG_2 \cong F_q^2 \oplus F_q^2 \oplus M_4(F_q)^2 \oplus M_3(F_q^2)^4.
\]

\[\square\]

3.3. **The group** \(G_3 = ((C_3 \times C_3) \times C_3) \times (C_2 \times C_2)\). **Group** \(G_3\) has the following presentation:
\[
G_3 = \langle x, y, z, w, t \mid x^2, [y, x], [z, x], [w, x]w^{-1}, [t, x]t^{-1}, y^2, [z, y]z^{-1}, [w, y], [t, y]t^{-1}, \]
\[z^3, [w, z]t^{-1}, [t, z], w^3, [t, w], t^3 \rangle.
\]

Further, \(G_3\) has 11 conjugacy classes as shown in the table below.

<table>
<thead>
<tr>
<th>rep</th>
<th>1</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>w</th>
<th>t</th>
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<th>xz</th>
<th>yw</th>
<th>zw</th>
<th>xyt</th>
</tr>
</thead>
<tbody>
<tr>
<td>size of class</td>
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<td>9</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>9</td>
<td>18</td>
<td>18</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>order of rep</td>
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<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

From above discussion, clearly the exponent of \(G_3\) is 6. Also \(G_3^\prime \cong (C_3 \times C_3) \times C_3\) with \(G_3/G_3^\prime \cong C_2 \times C_2\). Next, we discuss the unit group of \(F_qG_3\) when \(p > 3\).
Theorem 3.3. The unit group of $\mathbb{F}_qG_3$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:

$$U(\mathbb{F}_qG_3) \cong (\mathbb{F}_q^3)^4 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q)^2.$$ 

Proof. Since $\mathbb{F}_qG_3$ is semisimple, we have

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_q), \text{ for some } t \in \mathbb{Z}. \quad (3.14)$$

First assume that $k$ is even which means for any prime $p > 3$, $p^k \equiv 1 \mod 6$. This means $|S(\gamma_g)| = 1$ for each $g \in G_3$. Hence, equation (3.14), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q \oplus_{r=1}^{10} M_{n_r}(\mathbb{F}_q).$$

Incorporating Theorem 2.4 with $G'_3 \cong (C_3 \times C_3) \rtimes C_3$ in above to obtain

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{7} n_r^2. \quad (3.15)$$

Above equation gives us four possibilities $(2^4, 4, 6^2)$, $(2^3, 3^2, 5, 7)$, $(2, 3^2, 4^2, 5^2)$ and $(3^4, 4^2, 6)$ for the values of $n'_r$s. Further, consider the normal subgroup $H_3 = \langle t \rangle$ of $G_3$ having order 3 with $K_3 = G_3/H_3 \cong S_3 \times S_3$. It can be verified that $K_3$ has 9 conjugacy classes as shown in the table below.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$t$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$xy$</th>
<th>$xz$</th>
<th>$yw$</th>
<th>$zw$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>o</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Further, for all the representatives $g$ of $K_3$, $|S(\gamma_g)| = 1$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_qK_3 \cong \mathbb{F}_q \oplus_{r=1}^{5} M_{t_r}(\mathbb{F}_q), \text{ } t_r \in \mathbb{Z}.$$ 

Observe that $K_3/K'_3 \cong C_2 \times C_2$. So, above and Theorem 2.4 imply that

$$\mathbb{F}_qK_3 \cong \mathbb{F}_q^4 \oplus_{r=1}^{5} M_{t_r}(\mathbb{F}_q), \text{ with } 32 = \sum_{r=1}^{5} t_r^2, \text{ } t_r \geq 2.$$ 

This gives us the only choice $(2^4, 4)$ for values of $t'_r$s. Therefore, from Theorem 2.5 and equation (3.15), we conclude that $(2^4, 4, 6^2)$ is the correct choice for $n'_r$s which means

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q)^2. \quad (3.16)$$

Now we consider that $k$ is odd. We shall discuss this possibility in the following two cases:

Case 1: $p^k \equiv 1 \mod 6$. In this case, Wedderburn decomposition is given by equation (3.16).

Case 2. $p^k \equiv 5 \mod 6$. In this case, we have $S(\gamma_g) = \{\gamma_g\}$ for all the representatives $g$ of conjugacy classes. Therefore, Wedderburn decomposition is again given by equation (3.16). □
3.4. **The group** \( G_4 = C_2 \times ((C_3 \times C_3) \times C_3) \times C_2 \). Group \( G_4 \) has the following presentation:

\[
G_4 = \langle x, y, z, t \mid x^2, [y, x], [x, z]^{-1}, [w, x]w^{-1}, [t, x], y^2, [z, y], [w, y], [t, y]t^{-1}, z^3, [w, z]t^{-1}, [t, z], w^3, [t, w], t^3 \rangle.
\]

Further, \( G_4 \) has 20 conjugacy classes as shown in the table below.

<table>
<thead>
<tr>
<th>r</th>
<th>e</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>w</th>
<th>t</th>
<th>xy</th>
<th>xt</th>
<th>yz</th>
<th>wy</th>
<th>yt</th>
<th>zw</th>
<th>t^2</th>
<th>xyt</th>
<th>xt^2</th>
<th>yzt</th>
<th>z^2w</th>
<th>xyt^2</th>
<th>yz^2w</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<td>13</td>
<td>14</td>
<td>15</td>
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<td>18</td>
<td>19</td>
</tr>
<tr>
<td>o</td>
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<td>2</td>
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<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
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<td>10</td>
<td>11</td>
<td>12</td>
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<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
</tr>
</tbody>
</table>

From above discussion, clearly the exponent of \( G_4 \) is 6. Also \( G'_4 \cong (C_3 \times C_3) \times C_3 \) with \( G_4/\langle G'_4 \rangle \cong C_2 \times C_2 \). Next, we discuss the unit group of \( \mathbb{F}_q G_4 \) when \( p > 3 \).

**Theorem 3.4.** The unit group of \( \mathbb{F}_q G_4 \), for \( q = p^k \), \( p > 3 \) where \( \mathbb{F}_q \) is a finite field having \( q = p^k \) elements is as follows:

1. for any \( p \) and \( k \) even or \( p^k \equiv 1 \mod 6 \) with \( k \) odd, we have
   \[
   U(\mathbb{F}_q G_4) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^8.
   \]
2. for \( p^k \equiv 5 \mod 6 \) with \( k \) odd, we have
   \[
   U(\mathbb{F}_q G_4) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^4.
   \]

**Proof.** Since \( \mathbb{F}_q G_4 \) is semisimple, we have

\[
\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus \mathbb{F}_q^{t-1} M_{n_r}(\mathbb{F}_q), \quad \text{for some } t \in \mathbb{Z}. \tag{3.17}
\]

First assume that \( k \) is even which means for any prime \( p > 3 \), \( p^k \equiv 1 \mod 6 \). This means \( |S(\gamma g)| = 1 \) for each \( g \in G_4 \). Hence, equation (3.17), Theorems 2.1 and 2.2 imply that

\[
\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus \mathbb{F}_q^{19} M_{n_r}(\mathbb{F}_q).
\]

Using Theorem 2.4 with \( G'_4 \cong (C_3 \times C_3) \times C_3 \) in above to obtain

\[
\mathbb{F}_q G_4 \cong \mathbb{F}_q^{16} \oplus M_{n_r}(\mathbb{F}_q), \quad \text{where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{16} n_r^2. \tag{3.18}
\]

Above equation gives us the only possibility \((2^8, 3^8)\) for values of \( n_r \)s which means the required Wedderburn decomposition is

\[
\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^8. \tag{3.19}
\]

Now we consider that \( k \) is odd. We shall discuss this possibility in the following two cases:

Case 1: \( p^k \equiv 1 \mod 6 \). In this case, Wedderburn decomposition is given by equation (3.19).

Case 2. \( p^k \equiv 5 \mod 6 \). In this case, we have

\[
S(\gamma t) = \{\gamma t, \gamma t^2\}, \quad S(\gamma zt) = \{\gamma zt, \gamma zt^2\}, \quad S(\gamma yt) = \{\gamma yt, \gamma yt^2\}, \quad S(\gamma xyt) = \{\gamma xyt, \gamma xyt^2\}.
\]
and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representatives $g$ of conjugacy classes. Hence, equation (3.17), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^2}).$$

Above with Theorem 2.4 yields

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r}(\mathbb{F}_{q^2}),$$
where $n_r \geq 2$ with $104 = \sum_{r=1}^{8} n_r^2 + 2 \sum_{r=9}^{12} n_r^2$. (3.20)

Further, consider the normal subgroup $H_4 = \langle t \rangle$ of $G_4$ having order 3 with $K_4 = G_4/H_4 \cong C_2 \times ((C_3 \times C_3) \rtimes C_2)$. It can be verified that $K_4$ has 12 conjugacy classes as shown in the table below.

<table>
<thead>
<tr>
<th>r</th>
<th>e</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>w</th>
<th>xy</th>
<th>yz</th>
<th>yw</th>
<th>zw</th>
<th>z^2w</th>
<th>yz^2w</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>o</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

It can be seen that for all the representatives $g$ of $K_4$, $|S(\gamma_g)| = 1$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_qK_4 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{t_r}(\mathbb{F}_q), \ t_r \in \mathbb{Z}.$$ 

Observe that $K_4/K'_4 \cong C_2 \times C_2$. This with above and Theorem 2.4 imply that

$$\mathbb{F}_qK_4 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{t_r}(\mathbb{F}_q), \ with \ 32 = \sum_{r=1}^{8} t_r^2, \ t_r \geq 2.$$ 

This gives us the only choice $(2^8)$ for values of $t_r$'s. Therefore, Theorem 2.5 and equation (3.20) yields

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}), \ n_r \geq 2 \ with \ 36 = \sum_{r=1}^{4} n_r^2.$$ 

Above leaves us with the only choice $(3^4)$ for values of $n_r$'s which means the required Wedderburn decomposition is

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_{q^2})^4.$$ 

\[\Box\]

4. Discussion

We have characterized the unit groups of semisimple group algebras of 4 non-metabelian groups having order 108 and the results are verified using GAP. Clearly, the complexity in the calculation of Wedderburn decomposition upsurges with the increase in order of the group and we need to look into the Wedderburn decompositions of the quotient groups. The technique used for obtaining the Wedderburn decomposition works well provided the group has non-trivial normal subgroups of small order.
References