Design Optimization of Structures under Uncertainties

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Objectives

1. To develop a **simulation-based optimization approach** to perform the design optimization of the **stochastic systems**.
2. To develop simulation-based approaches to perform **stochastic design optimization** that includes methods:

a) Robust Design Optimization (RDO)

b) Reliability Based Design Optimization (RBDO).

c) The integration of the methods (a) and (b) resulting in **Reliability Based Robust Design Optimization (RBRDO).**

Achievements

For each objective, distinct approaches have been developed. The details are provided below:

1. Improved Stochastic Subset Optimization (iSSO) approach to perform design optimization of the stochastic systems. This approach, also applicable for deterministic optimization, features implementation of Voronoi tessellation.
2. An augmented formulation to perform RDO, where the design parameters are artificially considered as uncertain parameters. The mean and variance of the performance function are considered as measures of robustness.
3. A novel decoupled approach for performing RBDO that iteratively reducing the design space to identify the feasible region that meets reliability constraints. At each iteration Voronoi tessellation is implemented.
4. A first-of-its-kind stochastic simulation-based framework to perform the RBRDO. The proposed framework combines the developed concepts of RDO and RBDO.

The developed approaches outperform both **Taylor Series Expansion** based approximation approaches and **Sample Average Approximation** simulation-based approach in effectiveness and efficiency, respectively.

Publications

List of papers published in Journals:

1. M. A. Khalid, S. Bansal, and V. Ramamohan, "An augmented formulation for robust design optimization of structures using stochastic simulation method," *Research in Engineering Design*, vol. 34, no. 2, pp. 179–200, 2022, DOI: 10.1007/s00163-022-00405-z
2. M. A. Khalid and S. Bansal, "A Framework for Robust Design Optimization of Tuned Mass Dampers by Stochastic Subset Optimization," *International Journal of Structural Stability and Dynamics*, vol. 23, no. 13, pp. 2350155, 2023, DOI: 10.1142/S0219455423501559.
3. M. A. Khalid and S. Bansal, "Improved Stochastic Subset Optimization Method for Structural Design Optimization," *Advances in Engineering Software*, vol. 188, pp. 103568, 2024, DOI: 10.1016/j.advengsoft.2023.103568.
4. M. A. Khalid and S. Bansal, "Reliability-Based Design Optimization of Systems in Voronoi Tessellation-Based Partitioned Design Space."
5. M. A. Khalid and S. Bansal, "A framework for Reliability-Based Robust Design Optimization of stochastic systems."

No. of papers published in Conferences: 4Nos.

Methodologies

Improved Stochastic Subset Optimization (iSSO) Algorithm

Stochastic Engineering System

Performance function $\rightarrow g(\boldsymbol{\theta},\boldsymbol{\varphi})$

Design parameters $= \boldsymbol{\varphi} \in \Phi \subset \mathbb{R}^{n_\varphi}$

Uncertain variable $= \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^{n_\theta}$

At i+1th iteration, the design space is $I^{(i)} \in I^{(i-1)} \cdots \in I^{(0)} \in \Phi$.

Objective function $\rightarrow \underset{\boldsymbol{\varphi}\in\Phi}{\text{minimize:}} \quad \mathrm{E}_\theta[g(\boldsymbol{\theta},\boldsymbol{\varphi})]$

Augmented PDF $\rightarrow \pi(\boldsymbol{\theta},\boldsymbol{\varphi} \mid I^{(i)}) = \dfrac{g(\boldsymbol{\theta},\boldsymbol{\varphi})\, p(\boldsymbol{\theta},\boldsymbol{\varphi} \mid I^{(i)})}{\mathrm{E}_{\varphi,\theta}[g(\boldsymbol{\theta},\boldsymbol{\varphi})]}$

where $p(\boldsymbol{\theta},\boldsymbol{\varphi} \mid I^{(i)}) = p(\boldsymbol{\theta}|\boldsymbol{\varphi})\, p(\boldsymbol{\varphi} \mid I^{(i)})$

$\mathrm{E}_\theta[g(\boldsymbol{\varphi},\boldsymbol{\theta})] = \dfrac{\pi(\boldsymbol{\varphi} \mid I^{(i)})}{p(\boldsymbol{\varphi} \mid I^{(i)})}\mathrm{E}_{\varphi,\theta}[g(\boldsymbol{\varphi},\boldsymbol{\theta})]$

where $\pi(\boldsymbol{\varphi} \mid I^{(i)}) = \int_{I^{(i)}} \pi(\boldsymbol{\varphi},\boldsymbol{\theta})\, d\boldsymbol{\theta}$

$J(\boldsymbol{\varphi} \mid I^{(i)}) = \dfrac{\mathrm{E}_\theta[h_s(\boldsymbol{\varphi},\boldsymbol{\theta})]}{\mathrm{E}_{\varphi,\theta}[h_s(\boldsymbol{\varphi},\boldsymbol{\theta})]} = \dfrac{\pi(\boldsymbol{\varphi} \mid I^{(i)})}{p(\boldsymbol{\varphi} \mid I^{(i)})}$

Avoid direct evaluations of $\pi(\boldsymbol{\varphi} \mid I^{(i)})$. Instead, **simulate its samples** and use the information from them to **identify smaller subsets of the design space** likely to contain the optimal design solution.

The sensitivity of objective function $\mathrm{E}_\theta[g(\boldsymbol{\theta},\boldsymbol{\varphi})]$ to $\boldsymbol{\varphi}$ is determined by evaluating the average value (or equivalently volume density) of $J(\varphi)$ over any subset I_k, which is defined by $H(I)$:

$$H(I) = \frac{1}{V_I}\int_I J(\boldsymbol{\varphi} \mid I^{(i)})\, d\boldsymbol{\varphi} = \frac{1}{V_I}\int_I \frac{\pi(\boldsymbol{\varphi} \mid I^{(i)})}{p(\boldsymbol{\varphi} \mid I^{(i)})}\, d\boldsymbol{\varphi} = \frac{V_{I^{(i)}}}{V_I}\int_I \pi(\boldsymbol{\varphi} \mid I^{(i)})\, d\boldsymbol{\varphi}$$

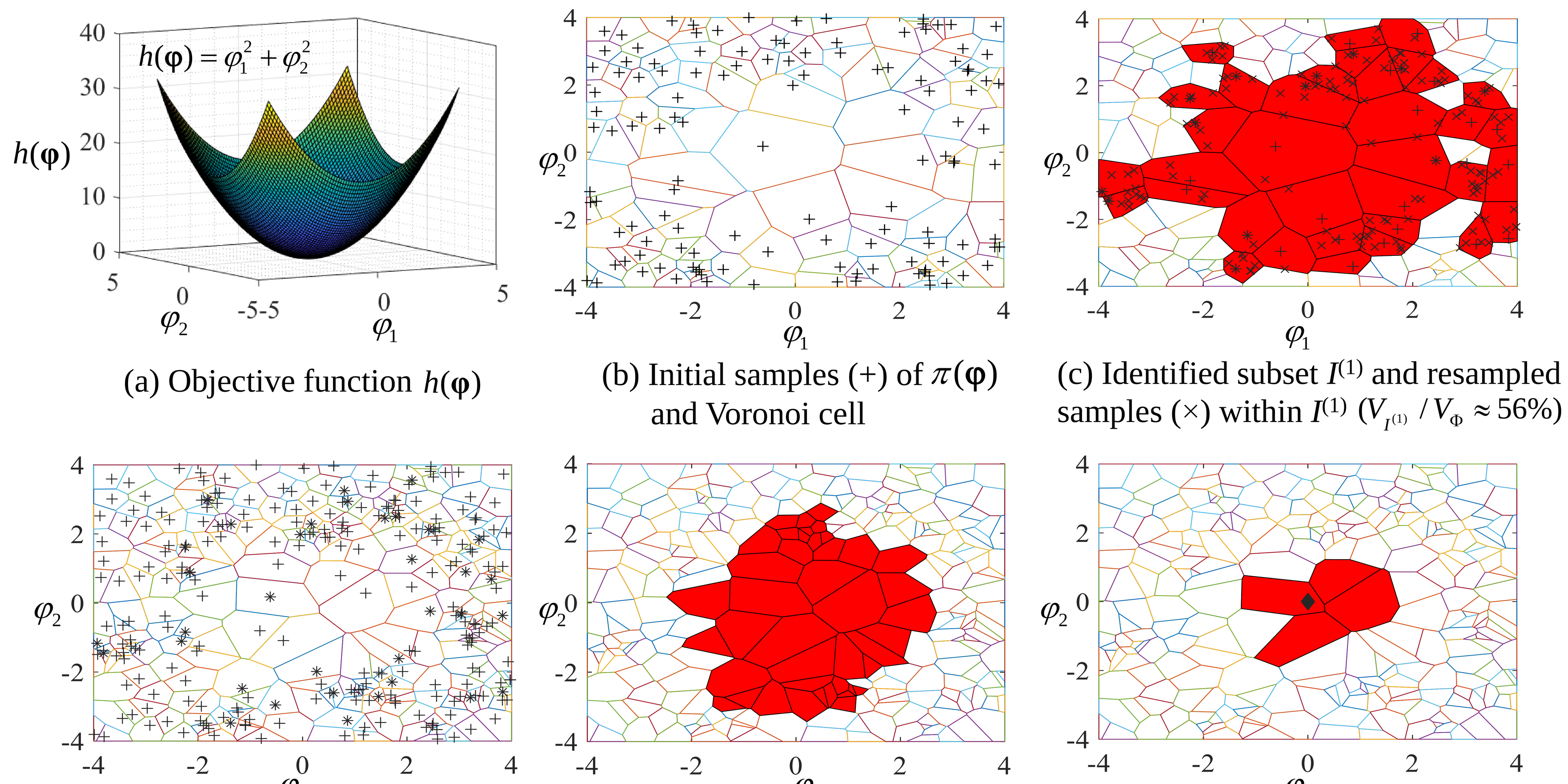
Divide the design space $I^{(i)}$ into unique n_s cells where the $v_k^{(i)}$ cell contains $\eta_k^{(i)}$ repeated samples, then estimate of $\pi(\boldsymbol{\varphi} \mid I^{(i)})$ is given as:

$$\bar{\pi}(\boldsymbol{\varphi} \mid I^{(i)}) = \frac{\eta_k^{(i)}}{N^{(i)} V_k^{(i)}} \geq 0, \qquad \forall \qquad \boldsymbol{\varphi} \in v_k^{(i)}$$

$$\bar{H}(I) = \frac{V^{(i)}}{N^{(i)}}\left[\frac{\eta_{(1)}^{(i)} + \eta_{(2)}^{(i)} + \cdots + \eta_{(S)}^{(i)}}{V_{(1)}^{(i)} + V_{(2)}^{(i)} + \cdots + V_{(S)}^{(i)}}\right]$$

$$I^{(i+1)} = \arg\min_{I \in A_\rho} \bar{H}(I)$$

$$A_\rho^{(i+1)} = \{I \subset I^{(i)} : \rho = N_I / N^{(i)}\}$$



(a) Objective function $h(\boldsymbol{\varphi})$ (b) Initial samples (+) of $\pi(\boldsymbol{\varphi})$ and Voronoi cell (c) Identified subset $I^{(1)}$ and resampled samples (×) within $I^{(1)}$ ($V_{I^{(1)}} / V_\Phi \approx 56\%$)

(d) Voronoi cells with combined samples of $\pi(\boldsymbol{\varphi})$ and $\pi(\boldsymbol{\varphi} \mid I^{(1)})$ (e) Identified subset $I^{(2)}$ ($V_{I^{(2)}} / V_\Phi \approx 29\%$) (f) Identified subset $I^{(5)}$ and optimal solution ($V_{I^{(5)}} / V_\Phi \approx 9\%$)

Fig. 1: Illustration of the proposed iSSO algorithm.

Identification of an Optimal Subset

A double-sort algorithm is proposed:

I. Sort the Voronoi cells in ascending order of the sample counts.

II. Cells with the same sample count in descending order of cell volume.

III. Finally, the initial cells and their corresponding samples, amounting to a total of $\rho N^{(i)}$ samples, are selected, providing an approximate optimal solution.

Augmented Formulation for Robust Design Optimization (RDO)

I. Augmented Formulation for Variance Minimization

$$\underset{\boldsymbol{\varphi}\in\Phi}{\text{minimize:}} \quad \sigma_g^2(\boldsymbol{\varphi}) = \mathrm{E}_\theta\left[\left(g(\boldsymbol{\theta},\boldsymbol{\varphi}) - \mu_g(\boldsymbol{\varphi})\right)^2\right] = \mathrm{E}_\theta\left[\left(g(\boldsymbol{\theta},\boldsymbol{\varphi}) - \mathrm{E}_\theta\left[g(\boldsymbol{\theta},\boldsymbol{\varphi})\right]\right)^2\right]$$

Modified structural performance function $h(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})$ is defined as follows:

$$h(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi}) = \left(g(\boldsymbol{\theta},\boldsymbol{\varphi}) - \bar{\varphi}\right)^2$$

$\underset{\boldsymbol{\varphi}\in\Phi,\bar{\varphi}\in\bar{\Phi}}{\text{minimize:}} \quad \mathrm{E}_\theta\left[h(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})\right] \Leftrightarrow \underset{\boldsymbol{\varphi}\in\Phi}{\text{minimize:}} \quad \sigma_g^2(\boldsymbol{\varphi})$

In the augmented space $\Omega \subset R^{n_\varphi + n_\theta + 1}$ of uncertain variables, an auxiliary PDF $\pi(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})$ is defined as:

$$\pi(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi}) = \frac{h(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})\, p(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})}{\mathrm{E}_{\theta,\varphi,\bar{\varphi}}[g(\boldsymbol{\theta},\boldsymbol{\varphi},\bar{\varphi})]}$$

II. Augmented Formulation for Mean Minimization

$$\underset{\boldsymbol{\varphi}\in\Phi}{\text{minimize:}} \quad \mu_g(\boldsymbol{\varphi}) = \mathrm{E}_\theta\left[g(\boldsymbol{\theta},\boldsymbol{\varphi})\right]$$

In the augmented space $\Omega \subset R^{n_\varphi + n_\theta}$ of uncertain variables, an auxiliary PDF $\pi(\boldsymbol{\theta},\boldsymbol{\varphi})$ is defined as:

$$\pi(\boldsymbol{\theta},\boldsymbol{\varphi}) = \frac{g(\boldsymbol{\theta},\boldsymbol{\varphi})\, p(\boldsymbol{\theta},\boldsymbol{\varphi})}{\mathrm{E}_{\theta,\varphi}[g(\boldsymbol{\theta},\boldsymbol{\varphi})]}$$

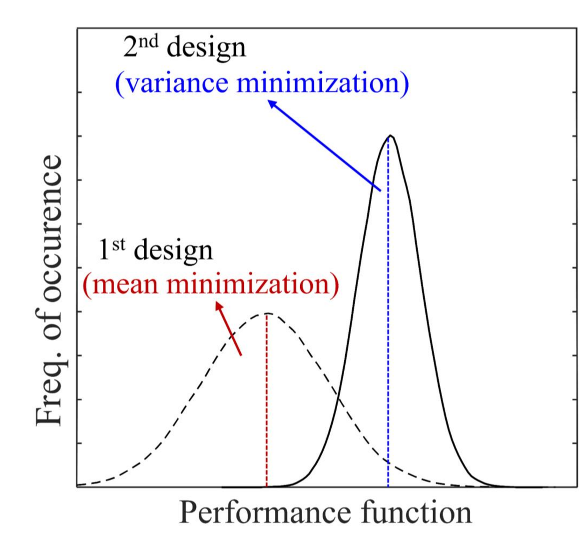


Fig. 2: Concept of RDO.

Mathematical Formulation of RDO

$$\text{Find} \quad [\boldsymbol{\varphi},\bar{\varphi}]^* = \arg\min_{\boldsymbol{\varphi}\in\Phi,\bar{\varphi}\in\bar{\Phi}}\left\{\alpha\frac{g(\boldsymbol{\theta},\boldsymbol{\varphi})}{\tilde{\mu}_g} + (1-\alpha)\frac{\left(g(\boldsymbol{\theta},\boldsymbol{\varphi}) - \bar{\varphi}\right)^2}{\tilde{\sigma}_g^2}\right\}$$

For Optimization perform "iSSO"

Reliability Based Design Optimization (RBDO) Algorithm

Mathematical formulation of RBDO:

$\underset{\boldsymbol{\varphi}\in\Phi}{\text{minimize:}} \quad h(\boldsymbol{\varphi})$

subject to: $\quad P_{F_j}(\boldsymbol{\varphi}) \leq P_{F_j}^*, \quad j = 1 \cdots R$

$\boldsymbol{\varphi}_L \leq \boldsymbol{\varphi} \leq \boldsymbol{\varphi}_U$

Augmented Failure Probability Function (FPF):

$$P_F(\boldsymbol{\varphi}) = P(F \mid \boldsymbol{\varphi}) = \frac{p(\boldsymbol{\varphi} \mid F) P(F)}{p(\boldsymbol{\varphi})} \propto p(\boldsymbol{\varphi} \mid F)$$

$$p(\boldsymbol{\varphi} \mid F) = \frac{p(\boldsymbol{\varphi}, F)}{P(F)} = \frac{\int_\Theta I_F(\boldsymbol{\varphi},\boldsymbol{\theta})\, p(\boldsymbol{\varphi})\, p(\boldsymbol{\theta})\, d\boldsymbol{\theta}}{P(F)}$$

Proposed approach starts with simulating failure samples following $p(\boldsymbol{\varphi} \mid F)$, which are used for its estimation and iteratively identification of feasible design space.

At the i+1th iteration ($i \geq 0$), the design subspace is $D_i \subset D_{i-1} \subset \cdots \subset D_0 = \Phi$.

Estimate of $p(\boldsymbol{\varphi} \mid F, D_i)$ by performing Voronoi tessellation as follows:

$$\bar{p}(\boldsymbol{\varphi} \mid F, D_i) = \frac{\eta_i^t}{N_0 V_i^t} \geq 0, \qquad \forall \qquad \boldsymbol{\varphi} \in v_i^t$$

An Intermediate feasible design space in D_i is defined as:

$\dfrac{P(D_{i+1} \mid F)}{P(D_i \mid F)} = p_0 \longrightarrow$ Subset Simulation concept

Ultimately, after M iterations, the entire design space is partitioned into M + 1 mutually exclusive and collectively exhaustive spaces:

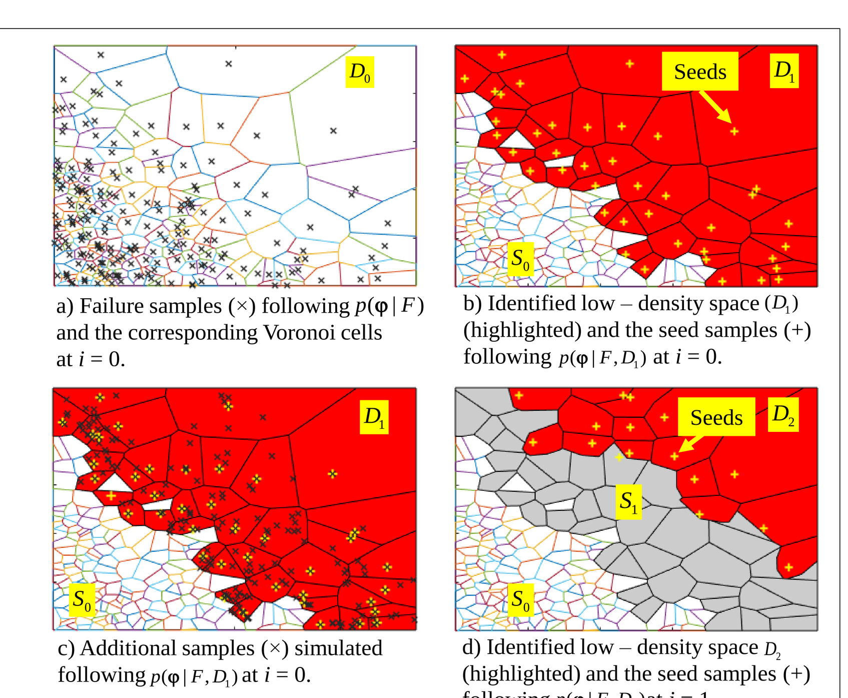
$\Phi = S_0 \cup \cdots \cup S_M$ where $S_i = D_i \cap \bar{D}_{i+1}$

Estimate of $p(\boldsymbol{\varphi} \mid F)$ is given as:

$$\bar{p}(\boldsymbol{\varphi} \mid F) = \sum_{i=0}^{M-1} p(\boldsymbol{\varphi} \mid F, S_i) P(S_i \mid F) + p(\boldsymbol{\varphi} \mid F, D_M) P(D_M \mid F)$$

$$P(S_i \mid F) = (1 - p_0)\left(p_0\right)^i$$

$$p(\boldsymbol{\varphi} \mid F, S_i) = \begin{cases} 0 & \boldsymbol{\varphi} \notin S_i \\ p(\boldsymbol{\varphi} \mid F, D_i) & \boldsymbol{\varphi} \in S_i \end{cases}$$

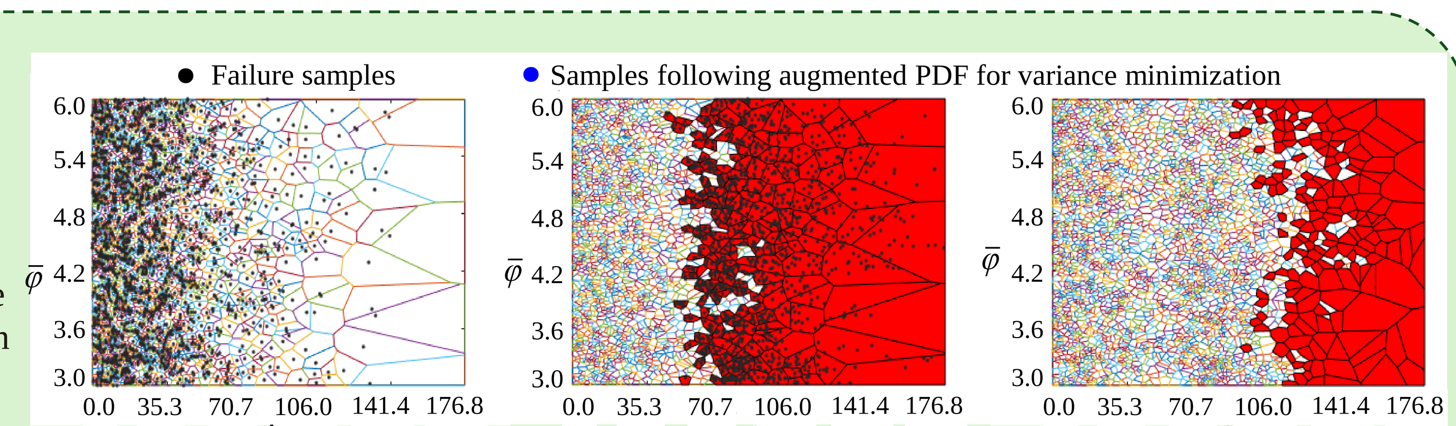


a) Failure samples (×) following $p(\boldsymbol{\varphi} \mid F)$ and the corresponding Voronoi cells at $i = 0$. b) Identified low – density space (D_i) (highlighted) and the seed samples (+) following $p(\boldsymbol{\varphi} \mid F, D_i)$ at $i = 0$. c) Additional samples (×) simulated following $p(\boldsymbol{\varphi} \mid F, D_i)$ at $i = 0$. d) Identified low – density space D_i (highlighted) and the seed samples (+) following $p(\boldsymbol{\varphi} \mid F, D_i)$ at $i = 1$.

Fig. 3: Proposed approach for identifying design subspace.

Design Optimization Results from RDO, RBDO, and RBRDO Approaches

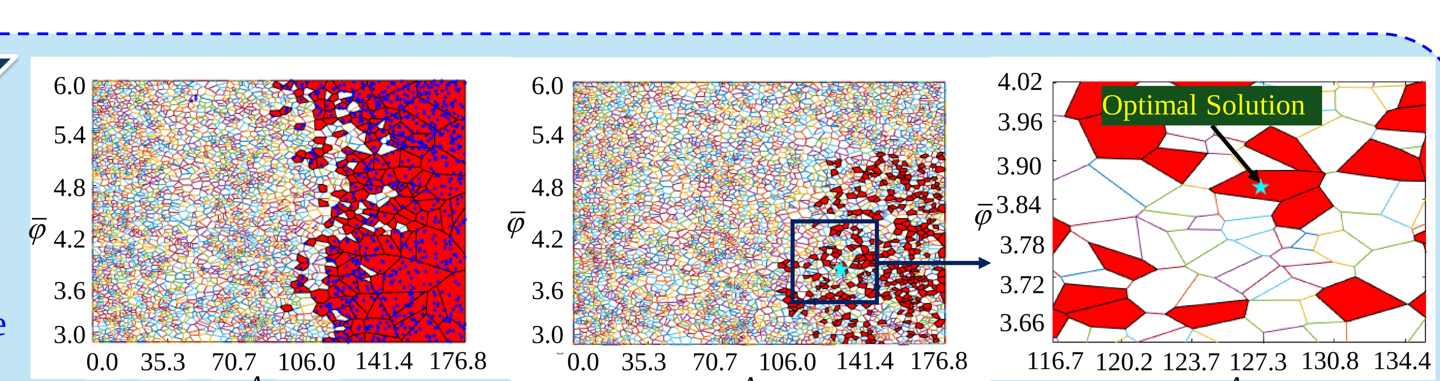
RBDO: Feasible design space identification



(a) Simulated failure samples and corresponding Voronoi cells at iteration – 1. (b) Identified intermediate failure region at iteration – 1 and resampling within it. (c) Identified feasible design space at iteration – 2 **(stopping criteria achieved)**

RBRDO

RDO in the feasible design space



(d) Simulated samples in the identified feasible design space at iSSO iteration – 1 (e) Identified subset at final iSSO iteration **(stopping criteria achieved)** (f) Zoomed-in version of (e)

Find: $[\boldsymbol{\varphi},\bar{\varphi}]^* = \arg\min_{\boldsymbol{\varphi}\in\Phi,\bar{\varphi}\in\bar{\Phi}}\left\{\alpha\dfrac{g(\boldsymbol{\theta},\boldsymbol{\varphi})}{\tilde{\mu}_g} + (1-\alpha)\dfrac{\left(g(\boldsymbol{\theta},\boldsymbol{\varphi}) - \bar{\varphi}\right)^2}{\tilde{\sigma}_g^2}\right\}$

Subjected to: $P_F(\boldsymbol{\varphi}) \leq 1\times10^{-3}$

$V \leq 500 cm^3$

$F = \{H(\boldsymbol{\theta},\boldsymbol{\varphi}) = S_i - 0.6 < 0\}, \ \forall\ i = 1,\ldots,4$

$g(\boldsymbol{\theta},\boldsymbol{\varphi}) =$ Displacement at free node of truss (u^*) when subjected to a static load P.

$\boldsymbol{\varphi} = [A_1, A_2], \boldsymbol{\theta} = [E_1, E_2, P], 0 \leq A_1 \leq 176.78$

$A_2 = 250 - \sqrt{2}A_1$ (cm)

$E_1 = \mathrm{N}(200,30)\ N/m^2, E_2 = \mathrm{N}(80,10)\ N/m^2, P = \mathrm{N}(100,20)\ N$

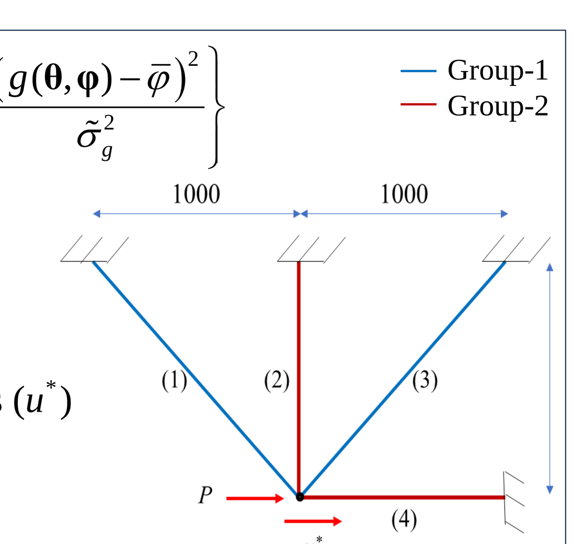


Fig. 4: Four bar truss.

Table 1: Comparison of results from different stochastic design approaches.

Design Method	**Optimal Design Parameters**		**Output**		
	A_1	A_2	Mean	Variance	Failure Probability
Reliability Based Design	176.8	0	4.100	1.133	**0.001**
Robust Design	105.0	101.50	4.399	**1.020**	0.008
Reliability Based Robust Design	125.4	72.6	**4.322**	**1.025**	**0.001**