EEL-306.

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Lecture

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We have seen that the joint-probability density function of samples from a Gaussian process depend only upon the mean values and the covariance matrix.

i.e., for any $n$ and any $t_1, \dots, t_n$ the joint p.d.f. of $(X(t_1), \dots, X(t_n))$ is given by

$$f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|K|}} e^{-\frac{(\underline{x}-\underline{\mu}_X)^T K^{-1}(\underline{x}-\underline{\mu}_X)}{2}}$$

where $\underline{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{\mu}_X \triangleq \begin{bmatrix} E(X(t_1)) \\ E(X(t_2)) \\ \vdots \\ E(X(t_n)) \end{bmatrix}$,

and the covariance matrix

$$K \triangleq E\left[(\underline{X} - \underline{\mu}_X)(\underline{X} - \underline{\mu}_X)^T\right]$$, $|K|$ is the determinant of $K$.

$$\underline{X} = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{bmatrix}.$$

Note that $\underline{X}$ and $\underline{x}$ are different. $\underline{X}$ is the vector of random variable $X(t_1), \dots, X(t_n)$ whereas $\underline{x}$ is the vector of values $(x_1, \dots, x_n)$ where we like to evaluate the p.d.f.

property 3 of Gaussian random processes -



If a Gaussian random process $X(t)$ is w.s.s. (wide sense stationary) then it is strictly stationary.

Proof: To show strict stationarity we will have to show that for any finite integer $n \geq 1$ and time instances $t_1, t_2, .., t_n$, and any $T$

$$f_{X(t_1), X(t_2), .., X(t_n)}(x_1, x_2, .., x_n) = f_{X(t_1+T), X(t_2+T), ..., X(t_n+T)}(x_1, x_2, .., x_n) \quad \text{—— (1)}$$

we can assume that $X(t)$ is w.s.s., i.e, for any two time instances $t_1$, and $t_2$

$$E(X(t_1)) = E(X(t_2)) \triangleq \mu \quad \text{—— (2.a)}$$

and

$$E(X(t_1)X(t_2)) = E(X(t_2-t_1)X(0))$$

$$\triangleq R(t_1-t_2) \quad \text{—— (2.b)}$$

(fn. of time difference only)

From (2.a) and (2.b) it is easy to see that

$$E\left[\left(X(t_1) - E(X(t_1))\right)\left(X(t_2) - E(X(t_2))\right)\right] \quad \text{—— (2.c)}$$

also depends only on the time difference $(t_2 - t_1)$

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We firstly note that for a W.S.S process $X(t)$

$$\underline{\mu_1} \triangleq \begin{bmatrix} E(X(t_1)) \\ \vdots \\ E(X(t_n)) \end{bmatrix} \text{ and}$$

$$\underline{\mu_2} \triangleq \begin{bmatrix} E(X(t_1+T)) \\ \vdots \\ E(X(t_n+T)) \end{bmatrix} \text{ are equal,}$$

and infact $\underline{\mu_1} = \underline{\mu_2} = \cancel{E(X(t))}\,\mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ — (3)

where $\mu \triangleq E(X(t))$

this follows from (2.a).

The $(i,j)^{th}$ entry of the covariance matrix of $(X(t_1), \ldots, X(t_n))$ is

$$K^{(1)}_{(i,j)} \triangleq E\Big[\big(X(t_i) - E(X(t_i))\big)\big(X(t_j) - E(X(t_j))\big)\Big]$$

$$= E\left[X(t_i)X(t_j)\right] - E(X(t_i))\,E(X(t_j))$$

$$= R(t_j - t_i) - \left[E(X(t))\right]^2$$

$$= R(t_j - t_i) - \mu^2. \quad \text{— (4.a)}$$

Similarly the $(i,j)^{th}$ entry of the covariance matrix of $(X(t_1+T), X(t_2+T), \ldots, X(t_n+T))$

is

$$K^{(2)}_{(i,j)} \triangleq E\Big[\big(X(t_i+T) - E(X(t_i+T))\big)\big(X(t_j+T) - E(X(t_j+T))\big)\Big]$$

$$= R\big((t_j+T) - (t_i+T)\big) - \mu^2$$

$$= R(t_j - t_i) - \mu^2 \quad \text{— (4.b)}$$

From (4.a) and (4.b) it follows that the matrices $K^{(1)}$ and $K^{(2)}$ are equal.

Also the mean vector $\mu_1$ and $\mu_2$ are equal (from (3)).

Since the p.d.f. only depends on the mean vector and the covariance matrix it follows that the p.d.f of $(X(t_1), .., X(t_n))$ and $(X(t_1+T), .., X(t_n+T))$ are the same.

Property 4

If $X(t_1)$ and $X(t_2)$ have a covariance of 0, i.e.,

$$E\left[\left(X(t_1) - E(X(t_1))\right)\left(X(t_2) - E(X(t_2))\right)\right] = 0,$$

then $X(t_1)$ and $X(t_2)$ are statistically independent.

Before proving property (4) we will discuss the following property of jointly Gaussian random variables.

FACT: If two jointly Gaussian random variables $X$ and $Y$ have zero covariance, i.e.,

$$E\left[(X-E(X))(Y-E(Y))\right]=0 ,$$ then

$X$ & $Y$ are statistically independent.

i.e., $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

Proof of this FACT: The joint p.d.f of $(X,Y)$ is given by

$$f_{X,Y}(x,y) = \frac{1}{(\sqrt{2\pi})^2\sqrt{|K|}}\, e^{-\frac{1}{2}\left\{\left[(x-E(X)) \;\; (y-E(Y))\right] K^{-1}\begin{bmatrix} x-E(X) \\ y-E(Y)\end{bmatrix}\right\}}$$

where
$$K \triangleq \begin{bmatrix} E[(X-E(X))^2] & E[(X-E(X))(Y-E(Y))] \\ E[(X-E(X))(Y-E(Y))] & E[(Y-E(Y))^2] \end{bmatrix}$$

Since the covariance between $X$ & $Y$ is zero, i.e.,

$$E\left[(X-E(X))(Y-E(Y))\right]=0,$$ it follows that

$K$ is diagonal i.e.,

$$\therefore K = \begin{bmatrix} E((X-E(X))^2] & 0 \\ 0 & E[(Y-E(Y))^2] \end{bmatrix} \quad (6)$$

$\therefore$ hence $|K| = E((X-E(X))^2)\,E((Y-E(Y))^2]$.

$\therefore$

$$f_{X,Y}(x,y) = \frac{1}{(\sqrt{2\pi})^2\sqrt{\sigma_X^2\sigma_Y^2}}\, e^{-\frac{g(x,y)}{2}}$$

where $\sigma_X^2 \triangleq \cancel{E(X)}\, E((X-E(X))^2)$ and

$\sigma_Y^2 \triangleq E((Y-E(Y))^2)$,

$$g(x,y) \triangleq [x-E(X) \quad y-E(Y)]\, K^{-1} \begin{bmatrix} x-E(X) \\ y-E(Y) \end{bmatrix}$$

$$= [x-E(X) \quad y-E(Y)] \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}^{-1} \begin{bmatrix} x-E(X) \\ y-E(Y) \end{bmatrix}$$

$$= \frac{(x-E(X))^2}{\sigma_X^2} + \frac{(y-E(Y))^2}{\sigma_Y^2}$$

$$\therefore f_{X,Y}(x,y) = \frac{1}{(2\pi)\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}}\, e^{-\left\{\frac{(x-E(X))^2}{2\sigma_X^2} + \frac{(y-E(Y))^2}{2\sigma_Y^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}}\, e^{-\frac{(x-E(X))^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_Y^2}}\, e^{-\frac{(y-E(Y))^2}{2\sigma_Y^2}}$$

$$= f_X(x)\, f_Y(y).$$

Property 4 now following from the above fact by using the substitution

$$X \triangleq X(t_1), \quad Y \triangleq X(t_2)$$

and the fact that $X(t_1), X(t_2)$ are jointly Gaussian.