However, just as in the case of Fourier series the mere existence of $p(f)$ does not ensure that the right-hand side of (4.5.7) will converge to $X(t)$. For this to hold we require further conditions on the "good behaviour" of $X(t)$, such as, e.g., that $X(s)$ *be of bounded variation in an interval containing the point* $s=t$. It may then be shown (Titchmarsh (1948)) that the right-hand side of (4.5.7) converges to $\frac{1}{2}\{X(t+0)-X(t-0)\}$, or more simply, converges to $X(t)$ at all continuity points (cf. "Jordan's Test" for Fourier series discussed in Section 4.2).

When the above conditions are satisfied (4.5.7) provides a representation for $X(t)$ as the limiting form of a "sum" of sine and cosine functions—but the crucial point is that here the *representation involves a continuous range of frequencies*, i.e. all values of f (from $-\infty$ to $+\infty$) are present in the integral in (4.5.7). This may be contrasted with the Fourier series representation (4.3.1) for general periodic functions in which only the *discrete set* of frequencies $(\ldots f_{-1}, f_0, f_1, \ldots)$ play any part.

We may now rewrite (4.5.7), (4.5.8) in terms of "angular frequency" by changing the variable in (4.5.7) from f to $\omega=2\pi f$. We then obtain,

$$X(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\,e^{i\omega t}\,d\omega, \tag{4.5.10}$$

where

$$G(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t)\,e^{-i\omega t}\,dt. \tag{4.5.11}$$

(In fact the functions $G(\omega)$, $p(f)$, are related simply by $G(\omega)\equiv\sqrt{2\pi}p(\omega/2\pi)$, and we have adopted this form of $G(\omega)$ so as to distribute the factor $(1/2\pi)$ equally between the two integrals.)

Parseval's relation

Corresponding to Parseval's relation for Fourier series (equation (4.4.2)) there is an analogous result for Fourier integrals, namely,

$$\int_{-\infty}^{\infty} X^2(t)\,dt=\int_{-\infty}^{\infty} |G(\omega)|^2\,d\omega. \tag{4.5.12}$$

To prove (4.5.12) we write,

$$\begin{aligned}\int_{-\infty}^{\infty} X^2(t)\,dt &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t)\left\{\int_{-\infty}^{\infty} G(\omega)\,e^{i\omega t}\,d\omega\right\}dt,\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\left\{\int_{-\infty}^{\infty} X(t)\,e^{i\omega t}\,dt\right\}d\omega,\end{aligned}$$