Transmission Bandwidth of FM Signals.

Lectures - 24 & 25

EEL-306.

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For a general message signal $m(t)$, we know that the FM signal is given by

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt\right)$$

$$= \mathrm{Re}\left(A_c\, e^{j2\pi k_f \int_0^t m(t)dt} \cdot e^{j2\pi f_c t}\right) \quad \text{--- (1)}$$

$\therefore$ the complex baseband representation of the passband signal $s_{FM}(t)$ is

$$\tilde{s}_{FM}(t) = A_c\, e^{j2\pi k_f \int_0^t m(t)dt}.$$

we also know that the Fourier transforms of $\tilde{s}_{FM}(t)$ and $s_{FM}(t)$ are related by

$$S_{FM}(f) = \frac{\tilde{S}^{*}_{FM}(-f-f_c) + \tilde{S}_{FM}(f-f_c)}{2} \quad \text{--- (2)}$$

(Note that $S_{FM}(f) = S^{*}_{FM}(-f)$ since $s_{FM}(t)$ is real-valued)

From equation (2) it is clear that if we know $\tilde{S}_{FM}(f)$ then we can find $S_{FM}(f)$ and from which we can know the transmission bandwidth of $S_{FM}(t)$.

Suppose that we know the Fourier transform of $m(t)$ (i.e $M(f)$).
Can we now find the Fourier transform of $\tilde{S}_{FM}(t) = A_c e^{j 2\pi k_f \int_0^t m(t)}$.

For any general $m(t)$, this is difficult, since $\tilde{S}_{FM}(t)$ is a non-linear function of $m(t)$. (Note that in the case of amplitude modulation (DSB-SC AM), the complex baseband AM signal $\tilde{S}_{AM}(t) = A_c m(t)$ is linearly related and therefore we had $\tilde{S}_{AM}(f) = A_c M(f)$, using which we could find $S_{AM}(f)$ and therefore the bandwidth of the DSB-SC signal).

$$\tilde{s}_{FM}(t) = A_c\, e^{j2\pi k_f \int_0^t m(t)\,dt}$$

$$= A_c\, e^{j2\pi k_f \int_0^t m(x)\,dx}$$

Assume that $\left|\int_0^t m(x)\,dx\right|$ is bounded for all t,

and using the Expansion of e^{jz}, i.e., for any $z \in \mathbb{C}$

$$e^{jz} = 1 + jz - \frac{z^2}{2!} - j\frac{z^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(jz)^n}{n!} \text{, we get}$$

$$e^{j2\pi k_f \int_0^t m(x)\,dx} = 1 + j2\pi k_f \int_0^t m(x)\,dx - \frac{1}{2!}\left(2\pi k_f \int_0^t m(x)\,dx\right)^2 \cdots$$

Since Fourier of $\int_0^t m(x)\,dx = \frac{M(f)}{j2\pi f}$ (assume $m(t)=0$ for $t<0$)

$$\text{Fourier}\left(e^{j2\pi k_f \int_0^t m(x)\,dt}\right)$$

$$= \delta(f) + \frac{k_f M(f)}{f} - \frac{1}{2}\left(G(f) \otimes G(f)\right) \cdots$$

where $G(f) \triangleq \frac{k_f M(f)}{f}$ has the same bandwidth as $M(f)$ and $\otimes$ denotes convolution.

$$\therefore \text{Fourier}\left(e^{j2\pi k_f \int_0^t m(x)\,dx}\right)$$

$$= \delta(f) + M(f) + \sum_{n=2}^{\infty} \frac{(j)^n}{n!} \underbrace{\left(G(f) \otimes \cdots \otimes G(f)\right)}_{n\text{-times convolution}} \quad \text{—} \quad (3)$$

Since $\overbrace{G(f) \circledast \cdots \circledast G(f)}^{n\text{-times convolution}}$ has a bandwidth which is n-times $M(f)$, one would expect ~~that~~ the bandwidth of $e^{jnk_f \int_0^t m(x)\,dx}$ is definitely more than the bandwidth of $M(f)$.

- Hence in general ~~the~~, for the same $M(f)$ the bandwidth of the corresponding FM signal is more than that of the corresponding AM signal.

However, we also note that the summation the R.H.S of (3) is from $n=2$ to ∞. Does this imply that the FM signal has <u>infinite</u> bandwidth. The answer is no, because the n-th term in the summation in (3) is also scaled down by $n!$.

Though we know that for the same (5)
$m(t)$, a FM signal occupies more bandwidth than an AM signal, we still cannot find an exact expression for the bandwidth of the FM signal for a general $m(t)$.
In the following we ~~thef~~ therefore focus our attention to the case of a single sinusoid $m(t)$, i.e.,

$$m(t) = A_m \cos 2\pi f_m t, \quad t \in [-T, T]$$

For this case,

$$\tilde{s}_{FM}(t) = A_c e^{j2\pi k_f \int_{-\infty}^{t} m(x)dx}$$

$$= A_c e^{j2\pi k_f \int_{-T}^{t} m(x)dx}$$

we assume $m(t)$ to be time-limited to $[-T, T]$, and is zero outside this interval.

$$= A_c e^{j2\pi k_f \left(\int_0^t m(x)dx + \int_{-T}^{0} m(x)dx \right)}$$

$$= A_c e^{j2\pi k_f \left(\frac{A_m}{2\pi f_m} \sin 2\pi f_m t + \phi_T \right)}$$

when $\phi_T \triangleq \int_{-T}^{0} m(x)dx$ is a constant (does not vary with t)

$$= A_c e^{j2\pi k_f \phi_T} e^{j \frac{k_f A_m}{f_m} \sin 2\pi f_m t}$$

$$t \in [-T, T] \quad (4)$$

Firstly note that

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$$\tilde{s}_{FM}(t) = A_c e^{j2\pi k_f \phi_T} e^{j\frac{k_f A_m}{f_m}\sin 2\pi f_m t}$$

is periodic with period $t_m = \frac{1}{f_m}$,

i.e., inside the interval $t \in [-T, T]$

$$\tilde{s}_{FM}(t+t_m) = \tilde{s}_{FM}(t + 1/f_m) = A_c e^{j2\pi k_f \phi_T} e^{j\frac{k_f A_m}{f_m}\sin 2\pi f_m (t+\frac{1}{f_m})}$$

$$= A_c e^{j2\pi k_f \phi_T} e^{j\frac{k_f A_m}{f_m}\sin 2\pi f_m t}$$

$$= \tilde{s}_{FM}(t) \quad \text{— (5)}$$

(assumes that both t and $t + 1/f_m \in [-T, T]$)

$\therefore$ for $t \in [-T, T]$, we can write the Fourier Series representation of $\tilde{s}_{FM}(t)$ by

$$\tilde{s}_{FM}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n \frac{t}{t_m}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}, \text{ where}$$

$$c_n \triangleq \frac{1}{t_m}\int_0^{t_m} \tilde{s}_{FM}(t) e^{-j2\pi n f_m t}\, dt$$

$$= \frac{1}{t_m}\int_0^{t_m} A_c e^{j2\pi k_f \phi_T} e^{j\frac{k_f A_m}{f_m}\sin 2\pi f_m t} e^{-j2\pi n f_m t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} A_c e^{j2\pi k_f \phi_T} e^{j\beta \sin v} e^{-jnv}\, dv \quad \Big| \begin{array}{l}\text{Substitution} \\ v = 2\pi f_m t\end{array}$$

(7)

$$c_n = \frac{A_c e^{j2\pi k_f \phi_T}}{} \frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta\sin v - nv)}\, dv$$

$$\therefore \quad c_n = A_c e^{j2\pi k_f \phi_T} J_n(\beta) \quad \text{, where}$$

$$J_n(\beta) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta\sin v - nv)}\, dv \qquad \text{— (6)}$$

is the n-th order Bessel function of the first kind.

Note that $\beta = \frac{k_f A_m}{f_m}$ is the modulation index.

~~Note that~~ ~~$J_0(\beta) =$~~ therefore for $t \in [-T, T]$,

$$\tilde{s}_{FM}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_m t}$$

$$= A_c e^{j2\pi k_f \phi_T} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m (t)} \qquad \text{— (7)}$$

Assuming $T \gg t_m$, we have

$$\tilde{S}_{FM}(f) \approx A_c e^{j2\pi k_f \phi_T} \sum_{n=-\infty}^{\infty} J_n(\beta)\, \delta(f - nf_m) \qquad \text{— (8)}$$

And therefore for $T \gg t_m$

$$s_{FM}(t) \approx A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(2\pi f_c t + 2\pi n f_m t + 2\pi k_f \phi_T)$$

$$\approx A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(2\pi (f_c + n f_m) t + 2\pi k_f \phi_T) \quad — (9)$$

i.e., even though $m(t)$ is a single sinusoid of frequency f_m, $s_{FM}(t)$ has multiple sinusoids at frequencies $(f_c + n f_m)$ where n is an integer.

However for any given β (fixed)

$$J_n(\beta) = \frac{1}{2\pi} \int_0^{2\pi} e^{j(\beta \sin v - nv)} \, dv$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(\beta \sin v - nv) \, dv$$

since $\sin(\beta \sin v - nv)$ is an odd function of v

$$\cancel{= \frac{1}{2\pi} \int_0^{2\pi} \cos \beta \sin v \cos nv \, dv + \frac{1}{2\pi} \int_0^{2\pi} \sin(\beta \sin v) \sin nv \, dv}$$

$$\therefore |J_n(\beta)| \le \frac{1}{2\pi} \int_0^{2\pi} |\cos(\beta \sin v - nv)| \, dv \le 1.$$

It is true that for any fixed β

$$\lim_{n\to\infty} J_n(\beta) = 0,$$ i.e.,

~~the~~ in (9) ~~the amplitude of~~

the component sinusoid of frequency $(f_c + nf_m)$

i.e., $A_c J_n(\beta)\cos(2\pi(f_c+nf_m)t + 2\pi k_f \phi_T)$

~~has~~ has an amplitude $A_c J_n(\beta)$ which

~~decreases~~ is small when n is large.

Note that $|J_n(\beta)| < 1$ for all n and all β, and therefore

$\cancel{J_0(\beta) = \frac{1}{2\pi}\int_0^{2\pi}\cos(\beta\sin x)\,dx \simeq 1}$, ~~therefore we~~

we would be <u>interested</u> in knowing those values of n for which $J_n(\beta)$ is significant (i.e., <u>not</u> very small compared to 1)

this is because in the summation

$$s_{FM}(t) \simeq A_c\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c+nf_m)t + 2\pi k_f\phi_T)$$

only those terms for which $J_n(\beta)$ is not very small compared to 1, will significantly contribute to the sum.

When $\beta \ll 1$,

then

$$J_0(\beta) \approx 1$$

$$J_1(\beta) \approx \frac{\beta}{2}, \text{ and}$$

$$J_n(\beta) \approx 0 \text{ for } n \geq 2.$$

Also $J_n(\beta) = (-1)^n J_{-n}(\beta)$

and therefore the only significant contributing terms in 9 correspond ~~to~~ $n = 0, +1, -1$,

$$\therefore s_{PM}(t) \approx A_c \cos(2\pi f_c t + \pi k_f \phi_T)$$

$$+ \frac{A_c\beta}{2} \cos(2\pi(f_c + f_m)t + \pi k_f \phi_T)$$

$$- \frac{A_c\beta}{2} \cos(2\pi(f_c - f_m)t + \pi k_f \phi_T)$$

$$\therefore S_{PM}(f) \approx \frac{A_c}{2} e^{-j\pi k_f \phi_T} \delta(f + f_c)$$

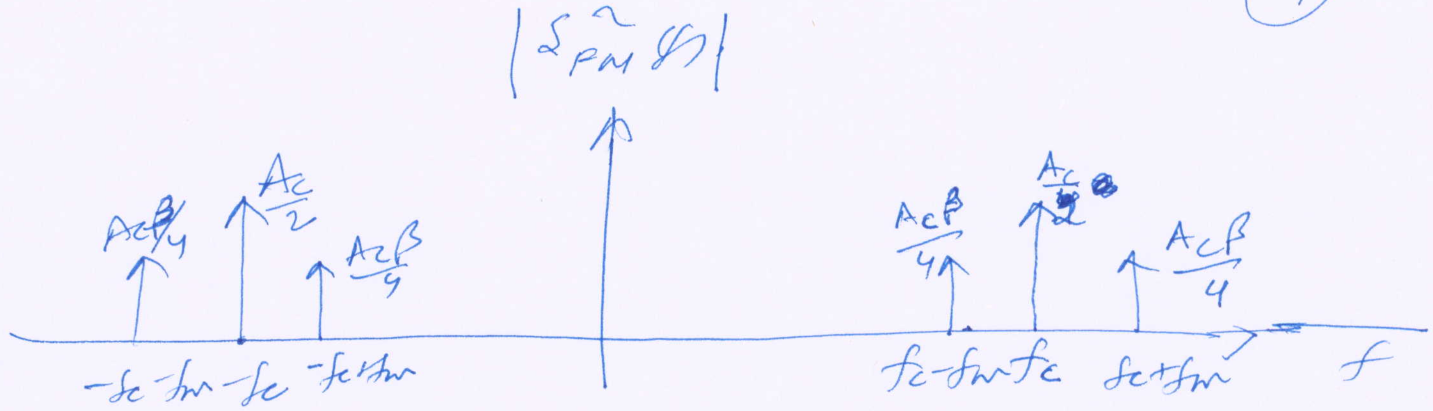
$$+ \frac{A_c}{2} e^{j\pi k_f \phi_T} \delta(f - f_c)$$

$$+ \frac{A_c\beta}{4} e^{-j\pi k_f \phi_T} \delta(f + f_c + f_m)$$

$$+ \frac{A_c\beta}{4} e^{+j\pi k_f \phi_T} \delta(f - f_c - f_m)$$

$$- \frac{A_c\beta}{4} e^{-j\pi k_f \phi_T} \delta(f + f_c - f_m)$$

$$- \frac{A_c\beta}{4} e^{j\pi k_f \phi_T} \delta(f - f_c + f_m)$$



Transmission bandwidth $= (f_c+f_m) - (f_c-f_m) = 2f_m$.

An FM signal with $\beta \ll 1$ is called the a <u>narrow band</u> FM signal since its bandwidth is only twice the ~~equal to~~ the bandwidth (~~almost same~~) of the input message signal $m(t)$.

Note that $\beta = \frac{k_f A_m}{f_m} = \frac{\Delta f}{f_m}$ where $\Delta f = k_f A_m$ is the maximum deviation of the instantaneous frequency of the FM signal from f_c.

$\therefore \quad 2f_m = \frac{2\Delta f}{\beta} \gg 2\Delta f \quad (\text{since } \beta \ll 1)$

i.e., the instantaneous bandwidth is much larger than $2\Delta f$ when the modulation index $\beta \ll 1$.

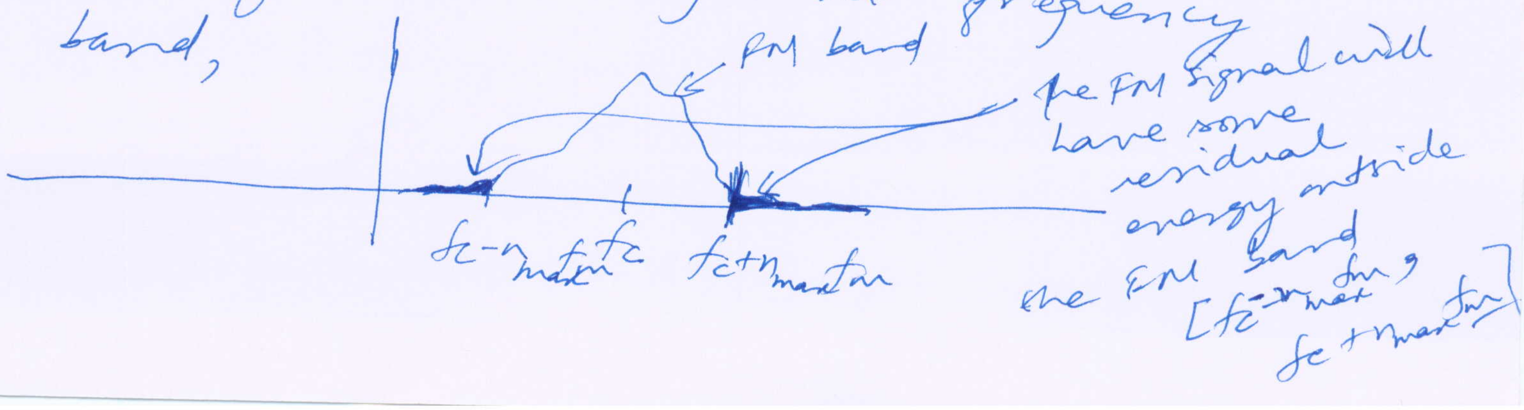
We next consider the case when $\beta \gg 1$. (12)

When $\beta \gg 1$, for a fixed β, let us define $n_{max}(\beta)$ as the largest integer n such that $|J_n(\beta)| > 0.01$ (i.e. 1 percent of the maximum possible value of 1).

$$n_{max}(\beta) \triangleq \max_{\left\{ \substack{n \geq 1 \\ n \in \mathbb{Z},\ |J_n(\beta)| > 0.01} \right\}} n$$

$\therefore$ the significant terms in the summation in (9) correspond to all terms from $n = -n_{max}(\beta) \ldots n = +n_{max}(\beta)$

(The choice of 1% is dependent upon the amount of interference that can be tolerated by another communication system operating in an adjacent frequency band,



$$\therefore s_{FM}(t) \approx \sum_{n=-n_{max}(\beta)}^{n_{max}(\beta)} A_c J_n(\beta) \cos(2\pi (f_c + n f_m)t + 2\pi k_f \phi_1) \quad (13)$$

Transmission bandwidth

$$\therefore \quad B_T \approx 2(n_{max}(\beta) - 1) f_m.$$

Table 4.1 in the book lists $n_{max}(\beta)$ for various values of β.

It is generally found that

$n_{max}(\beta) \propto \beta$ (i.e., with increasing β, $n_{max}(\beta)$ also increases)

$$\therefore \quad B_T \approx 2 \beta f_m = 2\Delta f \qquad \text{when } \beta >> 1$$

Since $\beta >> 1$, $B_T = 2\beta f_m >> 2f_m$.

Therefore in this case ($\beta >> 1$), the transmission bandwidth $B_T = 2\Delta f$ is <u>much larger</u> than the bandwidth ($2f_m$) of the input message signal $m(t)$. This is why an FM signal with $\beta >> 1$ is called a wideband FM signal.

CARSON'S RULE OF THUMB for the transmission bandwidth of FM signals.

For any β, the following rule of thumb is usually a good first-order estimate of the transmission bandwidth B_T.

$$B_T \simeq 2\Delta f + 2f_m . \quad \text{——} \quad (10)$$

APPROXIMATE TRANSMISSION BANDWIDTH OF GENERAL FM signals (not necessarily single frequency sinusoids)

let $A_m \triangleq \max_t |m(t)|$

$f_m \triangleq$ highest frequency component of $m(t)$.

$\Delta f \triangleq k_f A_m$. —— (11)

The approximate transmission bandwidth is given by

$$B_T \simeq 2\Delta f + 2f_m \quad \text{——} \quad (12)$$

where Δf and f_m are defined in (11).