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GENERATION & DEMODULATION OF FM SIGNALS. (Dr. SAIF K. MOHAMMED) (I.I.T. DELHI)

APRIL 9, 11, 2014 (LECTURES 26, 27)
(EEL 306)

GENERATION USING VOLTAGE CONTROLLED OSCILLATOR (VCO)

$$v(t) \longrightarrow \boxed{VCO} \longrightarrow v_{co}(t)$$

$$f_{vco}(t) = f_c + k_{vco}\, v(t)$$

instantaneous frequency of the vco o/p

$k_{vco} > 0$ is a constant

$\therefore$ Instantaneous phase of vco o/p is

$$\phi_{vco}(t) = \phi_{vco}(t=0) + 2\pi \int_0^t f_{vco}(t)\,dt$$

$$= \phi_{vco}(0) + 2\pi f_c t + 2\pi k_{vco} \int_0^t v(t)\,dt$$

$$\therefore\ v_{co}(t) = A_{vco} \cos\left[2\pi f_c t + 2\pi k_{vco} \int_0^t v(t)\,dt + \phi_{vco}(0)\right]$$

This is exactly an FM signal with $v(t)$ as the input message signal.

$\therefore \quad m(t) \longrightarrow \boxed{VCO} \longrightarrow S_{FM}(t).$

VCO can be used for FM signal generation.

COHERENT DEMODULATION OF FM SIGNAL USING PLL.

SYNCHRONIZATION PHASE.

Transmitter:

TX signal $\rightarrow S_{FM}(t) = A_1 \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt\right)$

During synch phase $m(t) = 0$
(only the unmodulated carrier is transmitted)

$\therefore \; S_{FM}(t) = A \cos 2\pi f_c t$ (during synch. phase)

Received signal $r(t) = G \, S_{FM}(t-\tau)$

G: amplitude loss, τ: delay

$\therefore \; r(t) = AG \cos 2\pi f_c (t-\tau).$

$= A_c \cos 2\pi f_c (t-\tau)$

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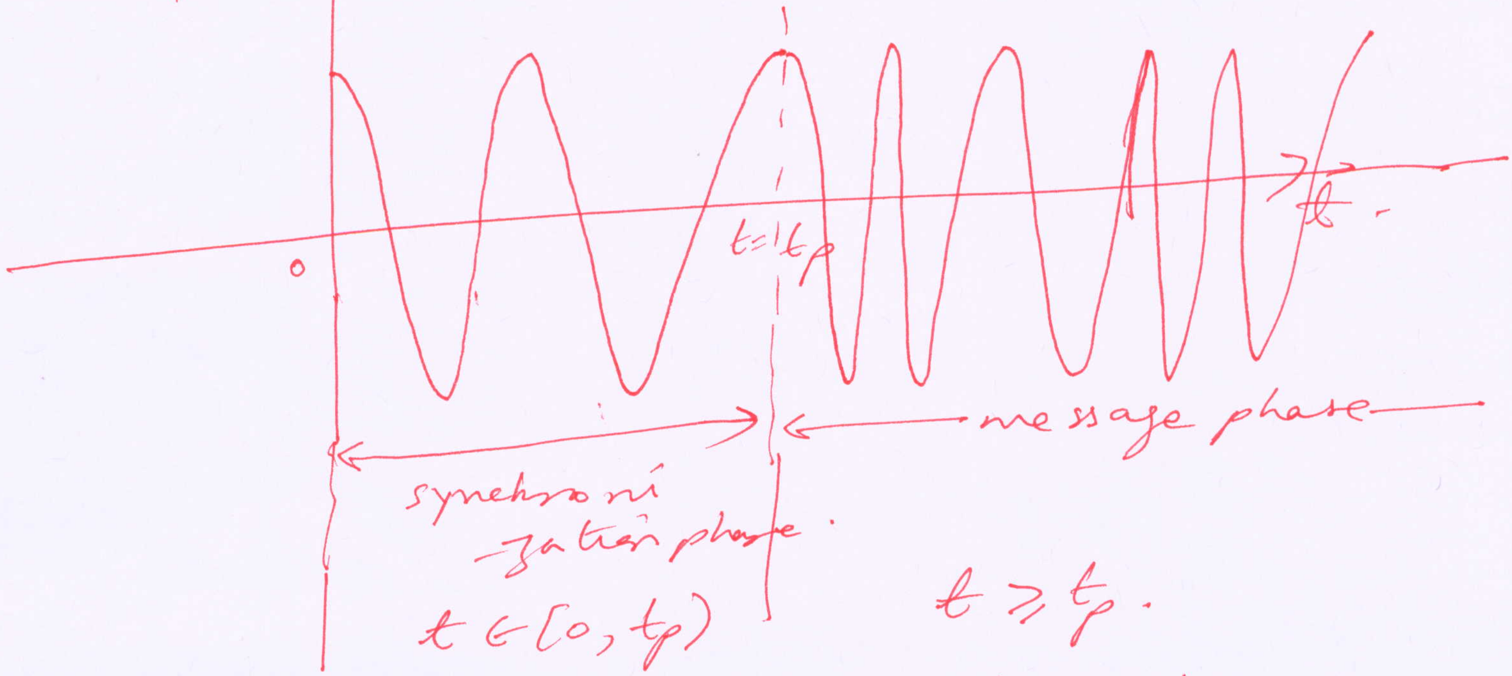
signal transmitted from TX.

$s_{FM}(t)$



unmodulated carrier

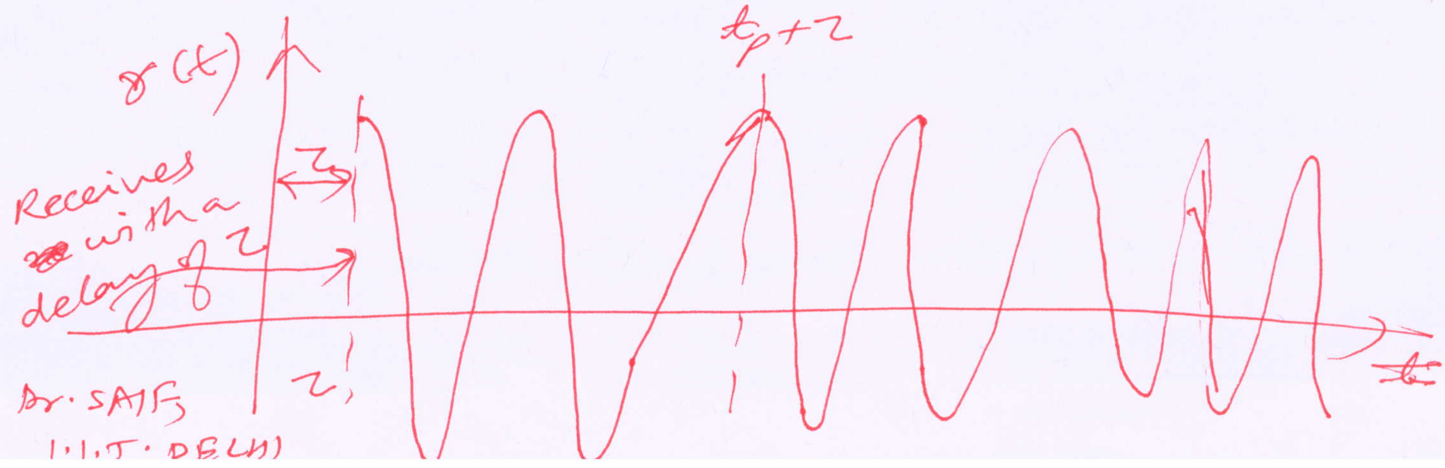
$$s_{FM}(t) = \begin{cases} A\cos 2\pi f_c t, & 0 \leq t < t_p \\ A\cos\left(2\pi f_c t + 2\pi k_f \int_{t_p}^{t} m(\alpha)\, d\alpha\right), & t \geq t_p \end{cases}$$

modulated carrier.

Received signal

$$r(t) = G\, s_{FM}(t-\tau).$$

AT THE RECEIVER.



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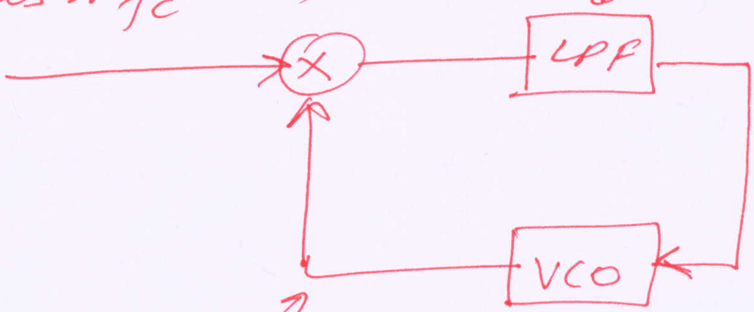
At the Receiver

SYNCHRONIZATION PHASE

$(\tau \le t < \tau + t_p)$

$r(t) = A_c \cos 2\pi f_c (t-\tau)$



the vco o/p at the end of the sync phase is

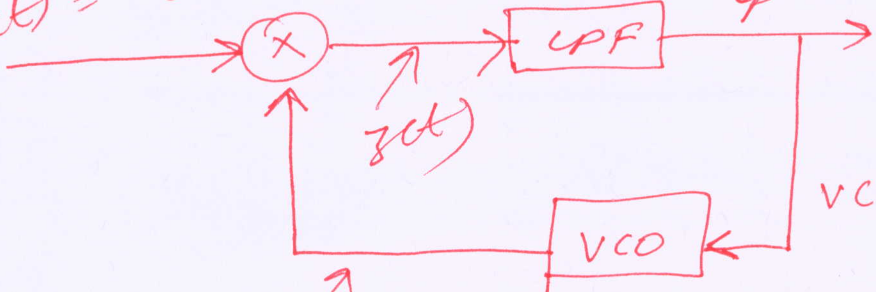
$$VCO(t) = A_v \cos\left(2\pi f_c (t-\tau) + \frac{\pi}{2}\right)$$

$$\Rightarrow -A_v \sin\left(2\pi f_c (t-\tau)\right)$$

MESSAGE PHASE

$(t \ge t_p + \tau)$

$$r(t) = A_c \cos\left(2\pi f_c (t-\tau) + 2\pi k_f \int_{t_p}^{t-\tau} m(x)\,dx\right)$$



$$-A_v \sin\left(2\pi f_c (t-\tau) + 2\pi k_{vco} \int_{t_p+\tau}^{t} v(t)\,dt\right)$$

Note that

$$r(t) = A_c \cos\left(2\pi f_c (t-\tau) + 2\pi k_f \int_{t_p+\tau}^{t} m(x-\tau)\,dx\right)$$

The input to the LPF is ?? (5)

- let $\phi_1(t) \triangleq 2\pi k_f \int_{t_p+\tau}^{t} m(x-\tau)\,dx$

and $\phi_2(t) \triangleq 2\pi k_{vco} \int_{t_p+\tau}^{t} v(x)\,dx$ ——→ (0)

Input to the LPF is

$$z(t) = -A_c A_v \cos(2\pi f_c(t-\tau)+\phi_1(t)) \sin(2\pi f_c(t-\tau)+\phi_2(t))$$

$$= \frac{-A_c A_v}{2}\left[\sin(\phi_2(t)-\phi_1(t)) + \sin(4\pi f_c(t-\tau)+\phi_1(t)+\phi_2(t))\right]$$

At the output of the LPF.

The signal $\sin(4\pi f_c(t-\tau)+\phi_1(t)+\phi_2(t))$ is filtered out (rejected) by the low pass filter.

∴ output of the low pass filter is

$$v(t) = -\frac{A_c A_v}{2}\sin(\phi_2(t)-\phi_1(t)) \quad \text{——→ (1)}$$

$$= -\frac{A_c A_v}{2}\sin(\phi_e(t)), \text{ where}$$

$\phi_e(t) \triangleq \phi_2(t)-\phi_1(t)$. ——→ (2)

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∴ we finally have for

$t \geq t_p + \tau$

differentiating (2) we get

$$\frac{d\phi_e(t)}{dt} = \frac{d\phi_2(t)}{dt} - \frac{d\phi_1(t)}{dt} \qquad (3)$$

$$= -2\pi k_f m(t-\tau) + 2\pi k_{vco} v(t)$$

Since $v(t) = -\frac{A_c A_v}{2} \sin(\phi_e(t))$ (from (1))

we finally get

$$\frac{d\phi_e(t)}{dt} = -\pi k_{vco} A_c A_v \sin \phi_e(t) - 2\pi k_f m(t-\tau) \qquad (4)$$

~~we firstly consider the simplest~~

One needs to solve (4) with the boundary condition that

$$\phi_e(t = t_p + \tau) = \phi_1(t_p+\tau) - \phi_2(t_p+\tau) \qquad (5)$$

$$= 0$$ (follows from equation 0)

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Note that (4) with initial condition (5)

is not a linear differential equation.

In general, it is difficult to solve (4)

Exactly Except in very simple cases,

~~for~~ for Example

$$m(t-\tau) = \begin{cases} A_m, & t \geq t_p + \tau \\ 0, & t < t_p + \tau. \end{cases}$$

i.e., a constant message signal

We next solve this special case

first. For this special case

$$\frac{d\varphi_e(t)}{dt} = -\pi k_{vco} A_c A_v \sin \varphi_e(t)$$
$$- 2\pi k_f A_m, \quad t \geq t_p + \tau$$

~~for~~ with $\varphi_e(t_p+\tau) = 0$. —— (6)

~~Note that equilibrium is only achieved when~~

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let:

$$a \triangleq \pi k_{vco} A_c A_v \text{ and}$$

$$b \triangleq 2\pi k_f A_m . \qquad (7)$$

we firstly consider the case when

$$b < a \quad (\text{note that both } a > 0 \text{ and } b > 0) \qquad (8)$$

for this case we have

$$-\frac{d\phi_e(t)}{dt} = a \sin\phi_e(t) + b .$$

$$\therefore \int_0^{\phi_e(t)} \frac{d\phi_e(t)}{a\sin\phi_e(t) + b} = -\int_{(\cancel{d}p+2)}^{t} dt \qquad (9)$$

we need to firstly solve the integral

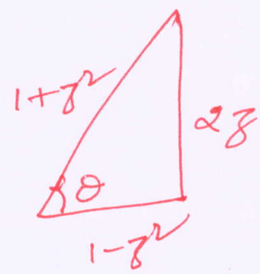
$$\int \frac{d\theta}{a\sin\theta + b} \quad \text{for} \quad (a > b).$$

let $z \triangleq \tan\frac{\theta}{2}$, then

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}$$

$$= \frac{2z}{1+z^2}$$



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$$\therefore \quad \theta = 2\tan^{-1} z$$

$$\Rightarrow \quad d\theta = \frac{2}{1+z^2}\, dz.$$

hence
$$\int \frac{d\theta}{a\sin\theta + b} = \int \frac{2\,dz}{(1+z^2)\left(a\cdot\frac{2z}{1+z^2} + b\right)}$$

$$= \int \frac{2\,dz}{\left(2az + b(1+z^2)\right)}$$

$$= \frac{2}{b}\int \frac{dz}{\left(1+z^2 + \frac{2a}{b}z\right)}$$

$$= \frac{2}{b}\int \frac{dz}{\left(\left(z+a/b\right)^2 - \left(\left(\frac{a}{b}\right)^2 - 1\right)\right)}$$

$$= \frac{2}{b}\int \frac{dz}{\left(z + a/b - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)\left(z + \frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)}$$

$$= \frac{1}{\sqrt{a^2-b^2}}\left[\int \frac{dz}{\left(z + \frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)} - \int \frac{dz}{\left(z + a/b + \sqrt{\left(a/b\right)^2 - 1}\right)}\right]$$

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~~with~~

10

$$\therefore \int \frac{d\theta}{a\sin\theta + b} = \frac{1}{\sqrt{a^2-b^2}} \ln\left[\frac{\tan\theta/2 + a/b - \sqrt{(a/b)^2-1}}{\tan\frac{\theta}{2} + \frac{a}{b} + \sqrt{(\frac{a}{b})^2-1}}\right] \quad \text{—} (10)$$

using (10) in (9) we get:

$$\ln\left[\frac{\left(\tan\frac{\varphi_e(t)}{2}\right) + \frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}}{\left(\tan\frac{\varphi_e(t)}{2}\right) + \frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}}\right]$$

$$= \ln\frac{\left(\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}\right)}{\left(\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}\right)} - t'\sqrt{a^2-b^2}.$$

where $t' \triangleq t - (t_p + \tau)$

$$\therefore \Rightarrow \frac{\tan\frac{\varphi_e(t)}{2} + \frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}}{\tan\frac{\varphi_e(t)}{2} + \frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}} = \left(\frac{\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}}{\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}}\right) e^{-t'\sqrt{a^2-b^2}}$$

$$\Rightarrow \tan\frac{\varphi_e(t)}{2} = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1} + \frac{2\sqrt{a^2/b^2-1}}{\left[1 - \left(\frac{\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}}{\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}}\right) e^{-t'\sqrt{a^2-b^2}}\right]}$$

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$$\therefore \phi_e(t) = 2\tan^{-1}\left\{ \frac{-a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1} + \frac{2\sqrt{\left(\frac{a}{b}\right)^2 - 1}}{\left(1 - \frac{\left(\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)}{\left(\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)} e^{-t\sqrt{a^2-b^2}}\right)} \right\} \quad \text{— (11)}$$

$$\therefore \lim_{t\to\infty} \phi_e(t) = 2\tan^{-1}\left\{ \frac{-a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1} \right\}. \quad \text{— (12)}$$

~~let~~ Since $a > b$, let

$b = a \sin \alpha$, then

$$\frac{-a}{b} + \sqrt{\frac{a^2}{b^2} - 1} = \frac{-1 + \cos\alpha}{\sin\alpha}$$

$$= -\frac{(1 - \cos\alpha)}{\sin\alpha}$$

$$= \frac{-2\sin^2 \alpha/2}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= -\tan\frac{\alpha}{2}.$$

$$\therefore \lim_{t\to\infty} \phi_e(t) = -\alpha = -\sin^{-1}(b/a). \quad \text{— (13)}$$

$$\therefore \lim_{t\to\infty} (a \sin\phi_e(t) + b) = 0.$$

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using (13)

$\therefore$ ~~[illegible]~~ in (6) we have

$$\lim_{t \to \infty} \frac{d\varphi_e(t)}{dt} = 0. \quad \text{—— (14)}$$

$\therefore$

$$\boxed{\text{At equilibrium:} \quad \frac{d\varphi_e(t)}{dt} = 0}$$

Since

$\frac{d\varphi_e(t)}{dt} = -(a \sin \varphi_e(t) + b)$, it turns

out that at equilibrium.

$$a \sin \varphi_e(t) + b = 0$$

$$\Rightarrow \quad \varphi_e(t) = \sin^{-1}(-b/a). \quad \text{—— (15)}$$

using (7) in (11) we can rewrite.

$$\varphi_e(t) = 2\tan^{-1}\left[\frac{-e - \sqrt{e^2-1} + 2\sqrt{e^2-1}}{\left[1 - \frac{(e-\sqrt{e^2-1})}{(e+\sqrt{e^2-1})} e^{-bt\sqrt{e^2-1}}\right.} \right. \quad \text{—— (16)}$$

where $e \triangleq \frac{a}{b} > 1$

$\therefore \quad e = \frac{k_{vco} A_C A_V}{2 k_f A_m} = \frac{A_C}{2} \cdot \frac{A_V k_{vco}}{\Delta f}$ where $\Delta f = k_f A$

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with $m(t-\tau) = A_m$ for $t \geq t_p + \tau$

the FM signal received is

$$r(t) = A_c \cos\left(2\pi f_c (t-\tau) + 2\pi k_f A_m \int_{t_p+\tau}^{t} dx\right)$$

$$= A_c \cos\left(2\pi f_c (t-\tau) + 2\pi k_f A_m (-t_p + (t-\tau))\right)$$

$$= A_c \cos\left(2\pi (f_c + k_f A_m)(t-\tau) - 2\pi k_f A_m t_p\right)$$

$$t \geq t_p + \tau.$$

$\therefore$ The FM signal is basically a sinusoid whose frequency is shifted from the carrier frequency by $\Delta f = k_f A_m$.

~~For such a scenario we have~~

~~using (12) and the fact that~~

$$\cancel{\lim_{t \to \infty} V(t) = -\frac{A_c A_v}{2} \sin(\phi_e(t))}$$

$$\cancel{= -\frac{A_c A_v}{2} \sin\left(2 \tan^{-1}\left\{ \frac{a}{b} + \sqrt{\frac{a^2}{b^2}}\right.\right.}$$

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Since
$$v(t) = -\frac{A_c}{2} A_v \sin\varphi_e(t),$$
using (11) we get.

$$\sin\varphi_e(t) = \frac{2x}{1+x^2}$$

where
$$x \triangleq -\frac{a}{b} - \sqrt{(a/b)^2-1} + \frac{2\sqrt{(a/b)^2-1}}{\left[1 - \frac{\left(a/b - \sqrt{(a/b)^2-1}\right)}{\left(a/b + \sqrt{(a/b)^2-1}\right)} e^{-t\sqrt{a^2-b^2}}\right]}$$

$$\therefore \lim_{t\to\infty} v(t) = -A_c A_v \frac{\left(-\frac{a}{b} + \sqrt{(a/b)^2-1}\right)}{\left(1 + \left(-a/b + \sqrt{(a/b)^2-1}\right)^2\right)}$$

$$= -A_c A_v \frac{\left(\sqrt{(a/b)^2-1} - a/b\right)}{\left[1 + \frac{a^2}{b^2} + (a/b)^2 - 1 - 2\frac{a}{b}\sqrt{(a/b)^2-1}\right]}$$

$$= \frac{+A_c A_v}{2(a/b)} = \frac{b}{2a}(A_c A_v)$$

$$= \frac{k_f A_m}{k_{vco} A_c A_v} \cdot A_c A_v$$

$$= \left(\frac{k_f}{k_{vco}}\right) A_m .$$

Impt.

$v(t)$ converges to $m(t)$ with a gain factor of (k_f/k_{vco}).

We have just seen that

if $\ell \triangleq \frac{a}{b}$ is > 1 then

$$\lim_{t \to \infty} v(t) = \left(\frac{k_f}{k_{vco}}\right) A_m .$$

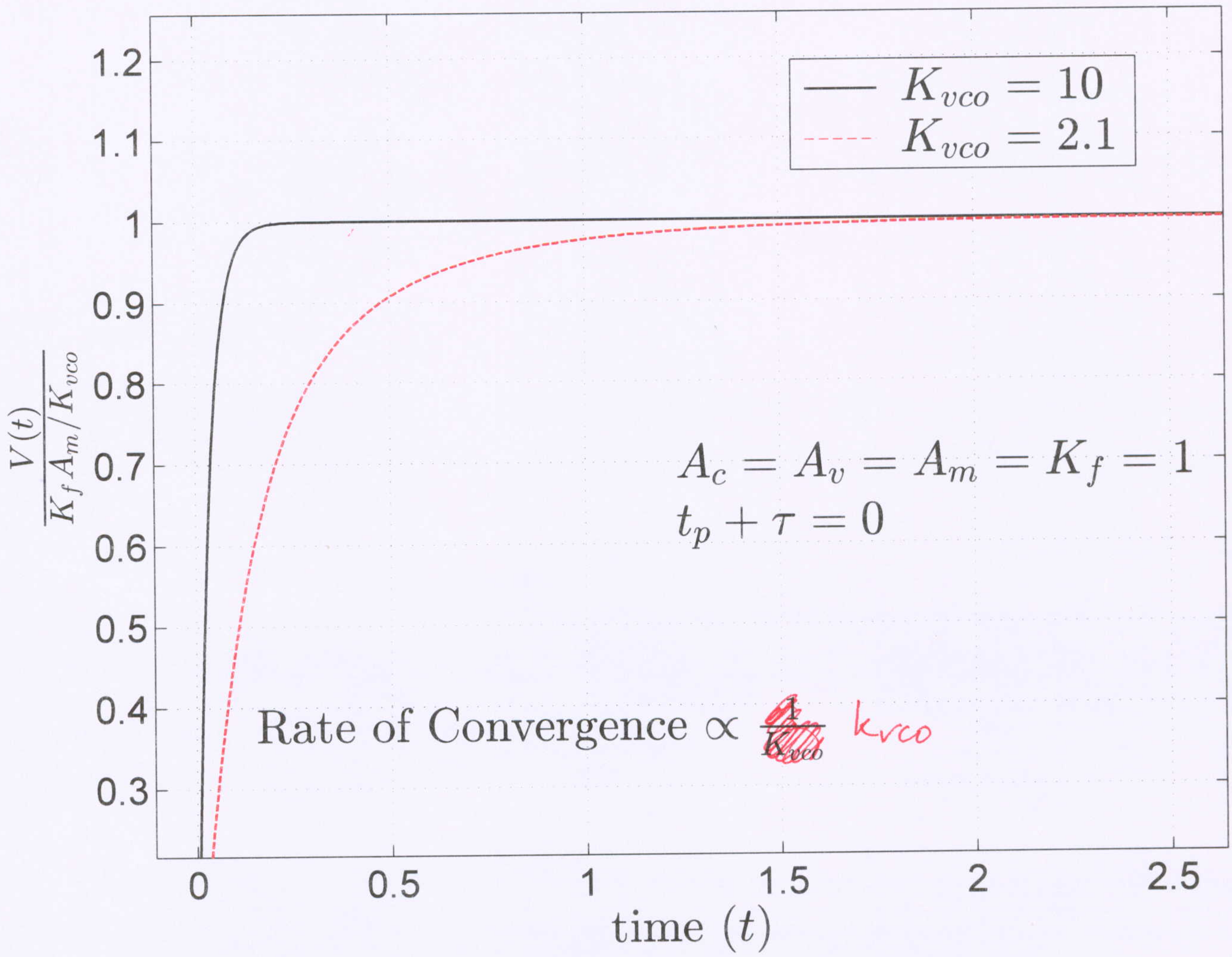
The next question is how fast is this convergence.

From (16) it is clear that for a given ℓ, a larger b implies faster convergence.

From (16) it is also clear that for a given b, a larger ℓ implies faster convergence.

i.e., a large $a = \pi k_{vco} A_c A_v$ implies faster convergence.

$\therefore$ for a given (fixed) A_c, k_f and A_m, faster convergence can be achieved by increasing k_{vco} (the frequency sensitivity of the VCO).



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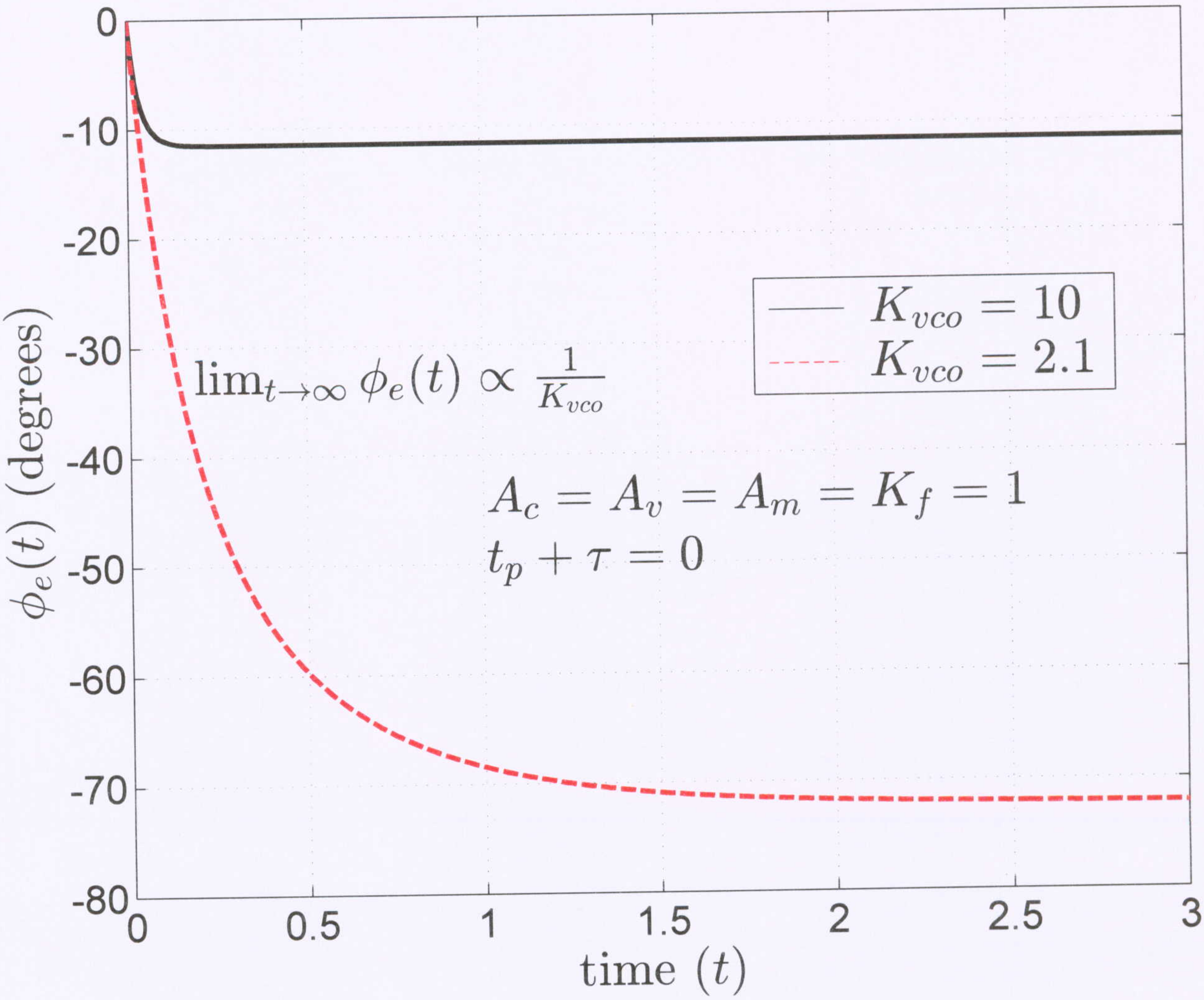

$\phi_e(t)$ is small when kvco is large.

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For large $\rho = \frac{a}{b}$,

$$\lim_{t\to\infty} \sin(\phi_e(t)) = -\frac{b}{a} \text{ is small.}$$

i.e., when $a >> b$

$$\lim_{t\to\infty} \phi_e(t) = -\sin^{-1}(b/a)$$

$$\approx -\frac{b}{a}. \quad \left(\text{Since for small } \theta, \ \sin\theta \approx \theta\right).$$

$\therefore$ Approximate analysis of

$$-\frac{d\phi_e(t)}{dt} = a\sin\phi_e(t) + b, \quad \phi_e(t = t_p+\tau) = 0.$$

↑ initial condition

Making the approximation

$$\sin(\phi_e(t)) \approx \phi_e(t)$$

$$-\frac{d\phi_e(t)}{dt} \approx a\phi_e(t) + b$$

$$\therefore \quad \frac{d\phi_e(t)}{a\phi_e(t)+b} \approx -dt$$

or $$\ln\left(1 + \frac{a}{b}\phi_e(t)\right) \approx -a(t - (t_p+\tau))$$

$\therefore \; \varphi_e(t) \approx \frac{b}{a}\left(e^{-a(t-(t_p+\tau))} - 1\right)$, and therefore

$V(t) \approx -\frac{A_c A_v}{2} \sin\left[\frac{b}{a}\left(e^{-a(t-t_p-\tau)} - 1\right)\right]$

Note that

$\lim_{t\to\infty} \varphi_e(t) \approx -b/a \approx -\sin^{-1}(b/a)$ 17

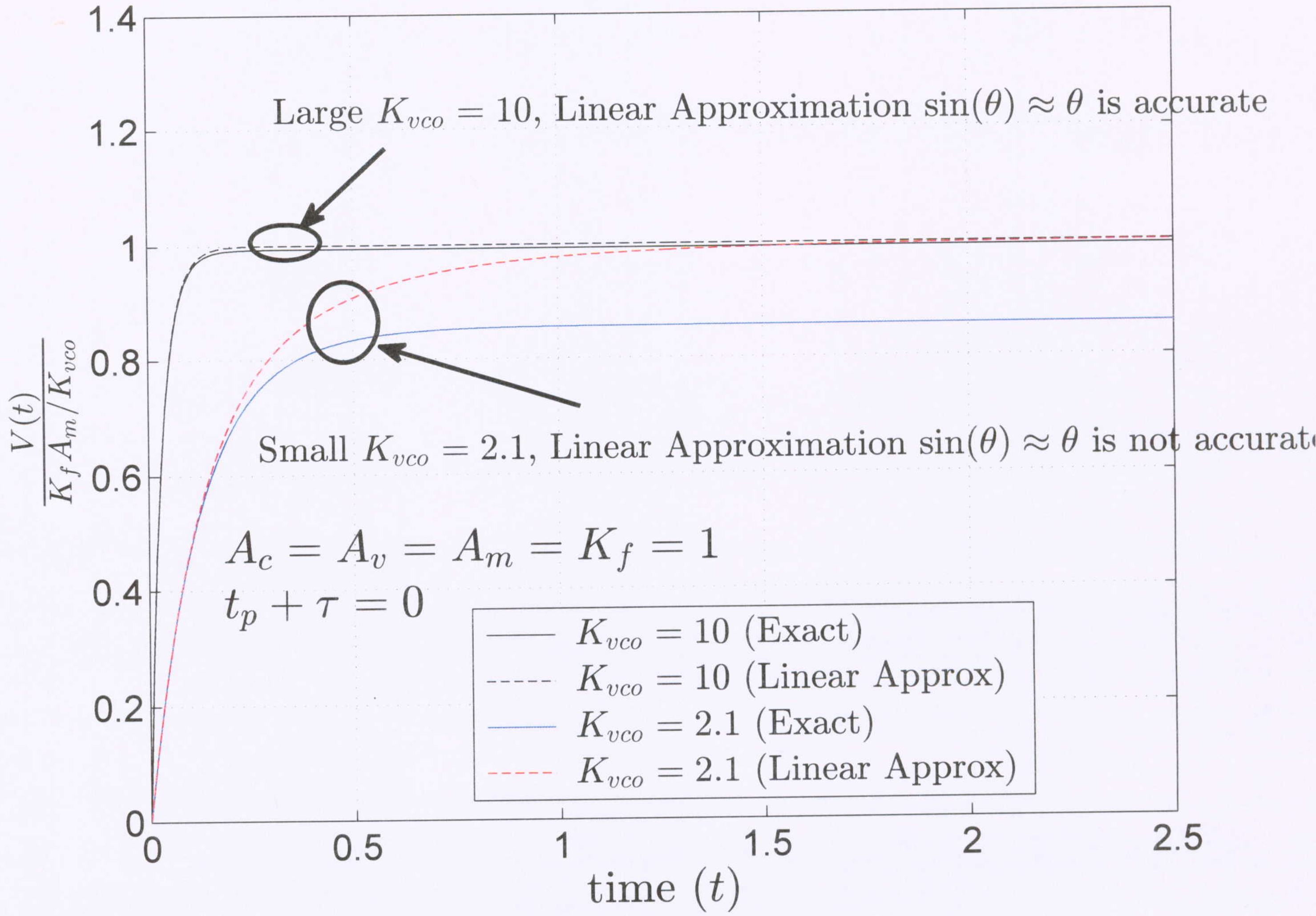
$\lim_{t\to\infty} V(t) \approx -\frac{A_c A_v}{2} \sin(-b/a)$

$= \frac{A_c A_v}{2} \sin(b/a)$

$\approx \frac{b}{2a} A_c A_v$

$= \left(\frac{k_f}{k_{vco}}\right) A_m.$

As Expected we will see that for large value of a/b (e.g. large k_{vco}), the linear approximation results (see 17) in accurate values for $V(t)$ other compared to the Exact value of $V(t)$ (see 16 for the Exact value of $\varphi_e(t)$).



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Summary.

For a constant-valued message signal.

$$m(t) = A_m , \qquad t \geq t_p .$$

the PLL at the receiver coherently demodulates the message signal from the received signal if and only if

$$\ell = \frac{a}{b} = \frac{k_{vco} A_c A_v}{2 k_f A_m}$$

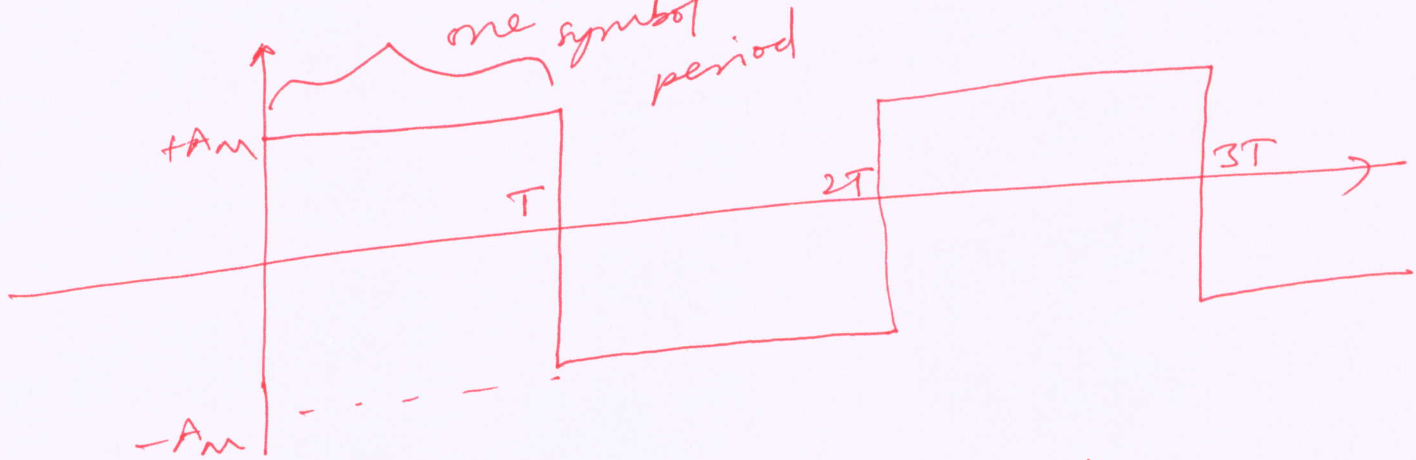
$$= \frac{A_c A_v k_{vco}}{2\Delta f} > 1 .$$

or $$\boxed{A_c A_v k_{vco} > 2\Delta f} .$$

~~Consider a signal m(t) which is not constant~~

Since $k_{vco} \propto$ rate of convergence, a message signal $m(t)$ having a larger amplitude should probably need a larger $A_v k_{vco}$ so that the PLL is able to lock.

Consider a binary $m(t)$



For the PLL to lock to this binary $m(t)$, in each symbol period, the PLL loop should be fast enough to converge, this requires that ~~and~~ (assuming $a/b \gg 1$) since

$$\phi_e(t) \approx \frac{b}{a}\left(e^{-a(t-(t_0+\tau))}-1\right) \quad \text{from (17)}$$

$$a \gg 1/T . \qquad \text{— (18)}$$

the ~~band~~ width of $m(t) \approx \frac{1}{T} \triangleq W$.

therefore the conditions are:

$A_c A_v k_{vco} > 2 k_f A_m$ and

$n\, k_{vco} A_c A_v \gg 1/T$.

Both these conditions are satisfied if

$$A_C A_V k_{VCO} >> B_T = 2\Delta f + 2f_m$$

transmission bandwidth of the FM signal.

thought questions.

Q.) What happens in the simple scenario where $m(t) = A_m$, if $a < b$?

Does the PLL loop still converge.

~~Q. What should be the bandwidth of the low pass filter (LPF) in the PLL so that the term $-\frac{A_c A_V}{2} \sin(\phi_2(t) - \phi_1(t))$ (see page 5 equation for $z(t)$) passes through the LPF.~~