EEL-306:

LECTURES- 28 & 29.

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Response of the PLL to a sinusoidal frequency deviation from the carrier frequency, and study of second order PLL's.

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In the previous lecture we had studied the response of a first order PLL to an input which had a constant frequency shift from the carrier frequency f_c (corresponds to $m(t) =$ step fn).

We had for $t \geq \tau + T_p$,

$$\frac{-d\phi_e(t)}{dt} = a \sin\phi_e(t) + 2\pi k_f m(t-\tau), \text{ where}$$

$$a \triangleq \pi k_{vco} A_c A_v, \text{ and we}$$

also had the initial condition $\phi_e(t=\tau+T_p)=0$.

In this lecture we will extend our analysis to the case where

$$m(t) = \begin{cases} A_m \cos 2\pi f_m (t-T_p), & t \geq T_p \\ 0, & t < T_p \end{cases}$$

with the assumption that

$\phi_e(t)$ is small, we have

$\sin \phi_e(t) \approx \phi_e(t)$ — ②, using this in ①

we get $$-\frac{d\phi_e(t)}{dt} = a\,\phi_e(t) + \cancel{b}\,k_f\, m(t-\tau), \quad t \geq \tau + T_p$$

with $\phi_e(t = \tau + T_p) = 0$. — ③.

Note that ③ is a linear differential equation, which can be solved.

If the solution to ③ gives us a $\phi_e(t)$ which is small for all $t > \tau + T_p$, then we can argue that our approximate analysis is correct (based on the fact that $\sin(\phi_e(t)) \approx \phi_e(t)$ for small $\phi_e(t)$).

We next try to solve ③.

Taking the Fourier transform of both sides of ③ we get

$$\cancel{-j2\pi f\, \phi_e(f) = a\,\phi_e(f) + b\,M(f)\, e^{-j2\pi f\tau}}$$

$$\cancel{\therefore\ \phi_e(f) = \frac{-M(f)\, e^{-j2\pi f\tau}}{(a + j2\pi f)}}$$

~~Taking~~ Since we are only interested in ~~on~~ $t \geq \tau + T_p$, we assume that $\phi_e(t) = 0$ for $t \leq \tau + T_p$. Now, taking the Fourier transform of both sides of (3) we get

$$-j2\pi f \phi_e(f) = a\phi_e(f) + 2\pi k_f M(f) e^{-j2\pi f\tau}$$

i.e., $$\phi_e(f) = \frac{-2\pi k_f M(f) e^{-j2\pi f \tau}}{(a + j2\pi f)} \quad - (4)$$

Also, since in the first order PLL

$$v(t) = -\frac{A_c A_V}{2} \sin \phi_e(t)$$
$$\simeq -\frac{A_c A_V}{2} \phi_e(t),$$ we have

$$V(f) \simeq -\frac{A_c A_V}{2} \phi_e(f)$$
$$= -\frac{A_c A_V M(f)}{2} \left(\frac{\phi_e(f)}{M(f)} \right) \quad \text{(using (4) we get)}$$
$$= \frac{A_c A_V M(f) 2\pi k_f e^{-j2\pi f\tau}}{2(a + j2\pi f)} \quad - (5)$$

To find $\varphi_e(t)$, we now take the inverse Fourier transform of (4).

Since $H(f) = \dfrac{1}{(a+j2\pi f)}$ corresponds to an LTI system with impulse response $h(t) = e^{-at}u(t)$,

from (4) we can easily say that

$$\varphi_e(t) = -2\pi k_f \left(m(t-\tau) \circledast e^{-at}u(t) \right)$$

convolution.

$$= -2\pi k_f \int_0^{t-\tau-T_p} e^{-ax}\, m(t-\tau-x)\, dx$$

$$(t \geq \tau + T_p)$$

This solution to (3) clearly satisfies the initial condition

$$\varphi_e(t = \tau + T_p) = 0.$$

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$$\therefore \varphi_e(t) = -2\pi k_f \int_0^{t-\tau-T_p} e^{-ax}\, m(t-\tau-x)\, dx$$

$t \geq \tau + T_p$.

which clearly satisfies the initial condition — (8)

$$\varphi_e(t = T_p + \tau) = 0.$$

with $m(t) = \begin{cases} A_m \cos 2\pi f_m(t - T_p) & t \geq T_p. \\ 0 & \text{for } t < T_p. \end{cases}$

we get

$$\therefore \varphi_e(t) = -2\pi k_f A_m \int_0^{t-\tau-T_p} e^{-ax} \cos 2\pi f_m(t-\tau-x-T_p)\, dx$$

$$= +2\pi k_f A_m \int_{t-\tau-T_p}^{0} e^{a(v-(t-\tau-T_p))} \cos 2\pi f_m v \, dv$$

$v \triangleq t-\tau-T_p-x$

$$\therefore \varphi_e(t) = -2\pi k_f A_m e^{-a(t-\tau-T_p)} \int_0^{t-\tau-T_p} e^{av} \cos 2\pi f_m v\, dv.$$

$t \geq \tau + T_p$.

using the fact that — (9)

$$\int e^{\alpha x} \cos \beta x \, dx = \frac{1}{\alpha^2+\beta^2} e^{\alpha x}\left[\alpha \cos \beta x + \beta \sin \beta x\right]$$

$$\therefore \int_0^{t-\tau-T_p} e^{av} \cos 2\pi f_m v \, dv$$

$$= \frac{1}{[a^2+(2\pi f_m)^2]} \Big\{ e^{a(t-\tau-T_p)} \big(a\cos 2\pi f_m (t-\tau-T_p) + 2\pi f_m \sin 2\pi f_m (t-\tau-T_p) \big) - a \Big\}$$

$$\therefore \phi_e(t) = \frac{-2\pi k_f A_m}{[a^2+(2\pi f_m)^2]} \Big\{ a\cos 2\pi f_m (t-\tau-T_p) + 2\pi f_m \sin 2\pi f_m (t-\tau-T_p) - a e^{-a(t-\tau-T_p)} \Big\} \quad \text{—(10)}$$

For all $t \geq \tau + T_p$.

$$|\phi_e(t)| = \frac{2\pi k_f A_m}{a} \left| \frac{a^2 \cos 2\pi f_m (t-\tau-T_p)}{(a^2+(2\pi f_m)^2)} + \frac{a \cdot 2\pi f_m \sin 2\pi f_m (t-\tau-T_p)}{(a^2+(2\pi f_m)^2)} - \frac{a^2}{(a^2+(2\pi f_m)^2)} e^{-a(t-\tau-T_p)} \right|$$

For any 2 real no's x & y, $\frac{x^2}{x^2+y^2} < 1$ and $\frac{|xy|}{x^2+y^2} < \frac{1}{2}$.

$$\therefore |\phi_e(t)| < \frac{2\pi k_f A_m}{a} \left(\frac{5}{2}\right)$$

Hence, a sufficient condition which guarantees that $|\phi_e(t)| \ll 1$ is that

$$a \gg 2\pi k_f A_m \text{ , i.e.,}$$

$$\text{i.e.} \quad \frac{k_{vco} A_r A_v}{2} \gg k_f A_m$$

for $a \gg 2\pi k_f A_m$

we know that $\varphi_e(t)$ is small,

$$\therefore \; v(t) \simeq -\frac{A_c A_v}{2} \varphi_e(t)$$

$$= \frac{A_c A_v \pi k_f A_m}{\left[a^2 + (2\pi f_m)^2\right]} \left\{ a \cos 2\pi f_m (t - \tau - T_p) + 2\pi f_m \sin 2\pi f_m (t - \tau - T_p) - a e^{-a(t - \tau - T_p)} \right\} \quad \text{— (13)}$$

In the steady state, i.e.,

when $(t - \tau - T_p) \gg 1/a$ — (14)

$$v(t) \simeq \frac{\pi A_c A_v k_f A_m}{\sqrt{a^2 + (2\pi f_m)^2}} \cos 2\pi f_m (t - \tau - T_p - \alpha)$$

where $\alpha = \tan^{-1}\left(\frac{2\pi f_m}{a}\right)$ — (15)

$\therefore$ Compared to $m(t) = A_m \cos 2\pi f_m (t - \tau - T_p)$,

$v(t)$ has an amplitude gain of

$$G(f_m) = \frac{\pi A_c A_v k_f}{\sqrt{a^2 + (2\pi f_m)^2}}$$ and a

phase lag of
$$\alpha = \tan^{-1}\left(\frac{2\pi f_m}{a}\right)$$ — (16)

Note that the gain factor can be written as

$$G(f_m) = \frac{k_f}{k_{vco}} \frac{a}{\sqrt{a^2+(2\pi f_m)^2}}$$

For minimum distortion we would like

$$G(f) = \frac{k_f}{k_{vco}} \cdot \frac{a}{\sqrt{a^2+(2\pi f)^2}}$$ to be almost

flat in the band where $m(t)$ has significant power; and also that the phase response

$$\alpha(f) = \tan^{-1}\left(\frac{2\pi f}{a}\right)$$ be almost

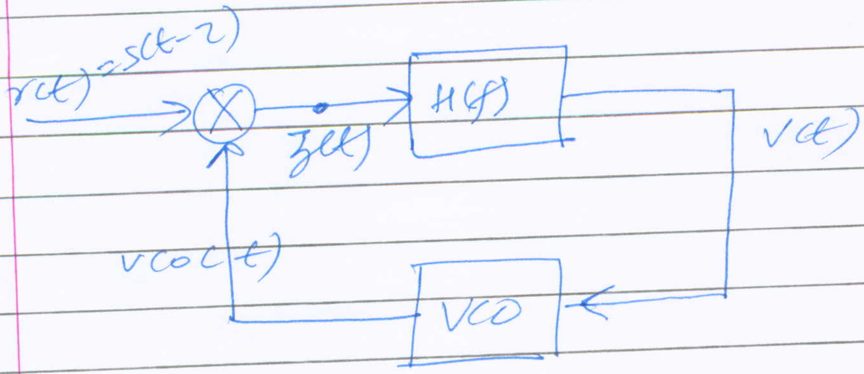
linear w.r.t f in the band ~~occupied~~ ~~by m~~ to which $m(t)$ is limited to.

These two requirements are met simultaneously if and only if

$$\frac{a}{2\pi} \gg \text{highest frequency component of } m(t)$$

Therefore we see that the same control parameter "a", controls both the bandwidth of $m(t)$ (such that the PLL reproduces it with minimal distortion) and also controls the maximum possible freq deviation ($k_f A_m$) such that $\phi_e(t)$ is small. (i.e., PLL is locked)

SECOND ORDER PHASE-LOCKED LOOP.



Compared to a first order system where $H(f) = 1$, in a second order system we have $H(f) = 1 + \frac{c}{jf}$ where c is a constant. — (18)

As before,

$$vco(t) = -A_v \sin\left(2\pi f_c (t-\tau) + 2\pi k_{vco} \int_{T_p+\tau}^{t} v(t)\, dt\right)$$

$$= -A_v \sin\left(2\pi f_c (t-\tau) + \phi_2(t)\right)$$

and

$$r(t) = s(t-\tau)$$

$$= A_c \cos\left(2\pi f_c (t-\tau) + 2\pi k_f \int_{T_p+\tau}^{t} m(x-\tau)\, dx\right)$$

$$= A_c \cos\left(2\pi f_c (t-\tau) + \phi_1(t)\right) \quad — (19)$$

The input to the low pass filter is

$$z(t) = -\frac{A_c A_v}{2} \sin\left(\phi_2(t) - \phi_1(t)\right) \quad — (20)$$

$$-\frac{A_c A_v}{2} \sin\left(4\pi f_c (t-\tau) + \phi_1(t) + \phi_2(t)\right).$$

this term gets rejected by the low pass filter.

∴ the output of the LPF is therefore

$$v(t) = -\frac{A_c A_V}{2}\left[\sin\varphi_e(t)\right] \circledast h(t) \quad -(21)$$

$\circledast$ denotes convolution

$h(t)$ is the impulse response of the LPF $(H(f))$

$$h(t) = \int_{-\infty}^{+\infty} H(f)\, e^{j2\pi f t}\, df$$

$$\varphi_e(t) = \varphi_2(t) - \varphi_1(t)$$

$$\therefore -\frac{d\varphi_e(t)}{dt} = 2\pi k_f\, m(t-\tau) + \pi k_{VCO} A_c A_V \left(\sin\varphi_e(t) \circledast h(t)\right) \quad -(22)$$

Assuming that the parameters k_{VCO}, and the filter $H(f)$ are such that $\varphi_e(t)$ is small then $\sin\varphi_e(t) \simeq \varphi_e(t)$

with this approximation in (22) we get

$$-\frac{d\varphi_e(t)}{dt} = 2\pi k_f\, m(t-\tau) + \pi k_{VCO} A_c A_V \left(\varphi_e(t) \circledast h(t)\right)$$

$$(t \geq T_p + \tau)$$

with the initial condition

$$\varphi_e(t) = 0 \text{ at } t = T_p + \tau. \quad -(23)$$

Taking Fourier transforms on both sides we get

$$-j2\pi f\, \varphi_e(f) = 2\pi k_f\, M(f)\, e^{-j2\pi f\tau} + \pi k_{VCO} A_c A_V\, \varphi_e(f)\, H(f)$$

$$\therefore \quad \frac{\phi_e(f)}{M(f)} = \frac{-2\pi k_f e^{-j2\pi f\tau}}{(j2\pi f + \pi k_{vco} A_c A_v H(f))} \quad \text{— (24)}$$

$$\therefore \quad \left|\frac{\phi_e(f)}{M(f)}\right| = \frac{2\pi k_f}{|j2\pi f + \pi k_{vco} A_c A_v H(f)|}$$

and

$$V(f) \approx -\frac{A_c A_v}{2}\left(\phi_e(t) \circledast h(t)\right) \quad \left(\sin(\phi_e(t)) \approx \phi_e(t)\right)$$

$$\therefore \quad V(f) = -\frac{A_c A_v}{2} \phi_e(f) H(f).$$

$$\therefore \quad \frac{V(f)}{M(f)} = -\frac{A_c A_v}{2} H(f) \frac{\phi_e(f)}{M(f)}$$

$$= \frac{\pi k_f A_c A_v H(f) e^{-j2\pi f\tau}}{(j2\pi f + \pi k_{vco} A_c A_v H(f))} \quad \text{— (25)}$$

with $H(f) = 1 + \frac{c}{jf}$, from (24) we have:

$$\frac{\phi_e(f)}{M(f)} = \frac{-2\pi k_f e^{-j2\pi f\tau}}{(j2\pi f + \pi k_{vco} A_c A_v (1 + c/jf))}$$

$$= \frac{(-2\pi k_f e^{-j2\pi f\tau})\, jf}{(-2\pi f^2 + \pi k_{vco} A_c A_v c + j\pi k_{vco} A_c A_v f)} \quad \text{— (26)}$$

$$\left(-2nf^2 + n k_{vco} A_z A_v C + j\, n k_{vco} A_z A_v f\right)$$

$$= \left(a_1 + j2nf\right)\left(b_1 + jf\right)$$

$$a_1 b_1 = n k_{vco} A_z A_v C$$

$$a_1 + 2n b_1 = n k_{vco} A_z A_v$$

i.e.:
$$a_1 + 2n \cdot \frac{n k_{vco} A_z A_v C}{a_1} = n k_{vco} A_z A_v$$

i.e.;
$$a_1^2 - \left(n k_{vco} A_z A_v\right) a_1 + 2n^2 k_{vco} A_z A_v C = 0$$

$$a_1 = \frac{n k_{vco} A_z A_v \pm \sqrt{n^2 k_{vco}^2 A_z^2 A_v^2 - 8n^2 k_{vco} A_z A_v C}}{2}$$

$$= \frac{n k_{vco} A_z A_v \pm n\sqrt{k_{vco} A_z A_v \left(k_{vco} A_z A_v - 8C\right)}}{2}$$

assuming $\left(k_{vco} A_z A_v > 8C\right)$ we set a_1 to be real valued and

$$a_1 = \frac{n}{2}\left(k_{vco} A_z A_v + \sqrt{k_{vco} A_z A_v \left(k_{vco} A_z A_v - 8C\right)}\right)$$

$$\therefore\ b_1 = \frac{n k_{vco} A_z A_v C}{a_1}$$

$$= \frac{2 k_{vco} A_z A_v C}{\left(k_{vco} A_z A_v + \sqrt{k_{vco} A_z A_v \left(k_{vco} A_z A_v - 8C\right)}\right)} \qquad (27)$$

$$= \frac{2C}{\left(1 + \sqrt{1 - \frac{8C}{k_{vco} A_z A_v}}\right)}$$

using the factorization on the previous page, for $k_{vco} A_c A_v > 8C$, we get
$\therefore$ In (26) we have

$$\frac{\phi_e(f)}{M(f)} = (-2\pi k_f)\, e^{-j\pi f \tau}\, \frac{jf}{(a_1 + j2\pi f)(b_1 + jf)}$$

where a_1, b_1 are given in (27).

$$\therefore \frac{-\phi_e(f)}{M(f)\, 2\pi k_f\, e^{-j\pi f\tau}} = \frac{jf}{(a_1 + j\pi f)(b_1 + jf)}$$

$$= \frac{d_1}{(a_1 + j2\pi f)} + \frac{d_2}{(b_1 + jf)}$$

$d_1 b_1 + d_2 a_1 = 0$ and

$d_1 + 2\pi d_2 = 1$

$\therefore \quad d_1 + 2\pi\left(-\frac{d_1 b_1}{a_1}\right) = 1.$

$\therefore d_1 = \frac{1}{(1 - 2\pi b_1/a_1)}$ and

$$d_2 = \frac{-b_1/a_1}{(1 - 2\pi b_1/a_1)} \qquad (28)$$

$$\therefore \frac{-\phi_e(f)}{2\pi k_f e^{-j\pi f\tau} M(f)} = \frac{d_1}{(a_1 + j\pi f)} + \frac{d_2}{(b_1 + jf)} = G(f).$$

Note that $g(t) = d_1 e^{-a_1 t} u(t)$
$\qquad + 2\pi d_2 e^{-2\pi b_1 t} u(t)$

$$\therefore \phi_e(t) = -2\pi k_f \left(m(t-\tau) \circledast g(t)\right) \qquad (29)$$

Note that under the condition that $k_{vco} A_c A_v > 8C$, it follows that $(a_1 > 2\pi b_1)$.

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$$\Phi_T(f) \simeq \frac{-\varphi_e(f)}{2\pi k_f M(f) e^{-j2\pi f\tau}}$$

$$= \frac{d_1}{a_1 + j2\pi f} + \frac{d_2}{b_1 + jf} \quad \text{—(34)}$$

Also from (23) we have

$$\frac{V(f)}{2\pi k_f M(f) e^{-j2\pi f\tau}} = \frac{A_c A_V}{2}\left(1 + \frac{c}{jf}\right)\left[\frac{d_1}{a_1 + j2\pi f} + \frac{d_2}{b_1 + jf}\right] \quad \text{—(35)}$$

Let us consider a sinusoidal $m(t) = \begin{cases} A_m \cos 2\pi f_m (t - T_p), & t \geq T_p \\ 0, & t < T_p. \end{cases}$

using (34) with the initial condition that $\varphi_e(t) = 0 \quad (t = T_p + \tau)$ and ~~inverting~~ inverting the Fourier transform in (34) we get

$$\varphi_e(t) = -2\pi k_f \Bigg[\frac{A_m d_1}{(a_1^2 + (2\pi f_m)^2)}\Big(a_1 \cos 2\pi f_m (t - \tau - T_p) + 2\pi f_m \sin(2\pi f_m (t - \tau - T_p)) - a_1 e^{-a_1 (t - \tau - T_p)}\Big)$$

$$+ \frac{A_m 2\pi d_2}{((2\pi b_1)^2 + (2\pi f_m)^2)}\Big(2\pi b_1 \cos 2\pi f_m (t - \tau - T_p) + 2\pi f_m \sin(2\pi f_m (t - \tau - T_p)) - 2\pi b_1 e^{-2\pi b_1 (t - \tau - T_p)}\Big)\Bigg] \quad \text{—(35)}$$

~~It can be shown that $|\varphi_e(t)| \ll 1$ for all $t \geq \tau + T_p$ if $f_m \ll \sqrt{\frac{k_{vco} A_c A_V C}{2}}$ and $k_{vco} A_c A_V \gg (2k_f + 2f_m$~~

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From eqn. (35) we have

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$$-\varphi_e(t) = \frac{2\pi k_f A_m d_1}{a_1}\left\{ \frac{a_1^2 \cos 2\pi f_m (t-\tau-T_p)}{a_1^2+(2\pi f_m)^2} + \frac{a_1 2\pi f_m \sin 2\pi f_m (t-\tau-T_p)}{a_1^2+(2\pi f_m)^2} - \frac{a_1^2}{a_1^2+(2\pi f_m)^2} e^{-a_1(t-\tau-T_p)} \right\}$$

$$+ 2\pi k_f \frac{(2\pi d_2) A_m}{(2\pi b_1)}\left\{ \frac{(2\pi b_1)^2 \cos 2\pi f_m (t-\tau-T_p)}{((2\pi b_1)^2+(2\pi f_m)^2)} + \frac{(2\pi b_1)(2\pi f_m) \sin 2\pi f_m (t-\tau-T_p)}{((2\pi b_1)^2+(2\pi f_m)^2)} - \frac{(2\pi b_1)^2}{((2\pi b_1)^2+(2\pi f_m)^2)} e^{-2\pi b_1(t-\tau-T_p)} \right\}$$

Since for any two ~~positive~~ real number x and y $\frac{x^2}{x^2+y^2} < 1$ and $\frac{|xy|}{x^2+y^2} < \frac{1}{2}$.

we have

$$|\varphi_e(t)| < \left(2\pi k_f A_m \left|\frac{d_1}{a_1}\right| + 2\pi k_f A_m \left|\frac{d_2}{b_1}\right|\right)\left(\frac{5}{2}\right).$$

Since $\frac{d_1}{a_1} + \frac{d_2}{b_1} = 0$, we further get

$$|\varphi_e(t)| < 2\pi k_f A_m \left|\frac{d_1}{a_1}\right| 5$$

$$= 5 \cdot \frac{2\pi k_f A_m}{|a_1 - 2\pi b_1|} = \frac{5 \cdot 2\pi k_f A_m}{(a_1 - 2\pi b_1)}$$ (since $a_1 > 2\pi b_1$ (see page 13 bottom))

∴ it follows that:

$|\varphi_e(s)|$ will be guaranteed

to be small if

$$(a_1 - 2\pi b_1) \gg 2\pi k_f A_m$$

$$\text{or} \quad a_1 \gg (2\pi k_f A_m + 2\pi b_1)$$

i.e., from the definition of a_1 and b_1

in (27), a further sufficiency

condition for $|\varphi_e(s)| \ll 1$ (small)

is that

$$\frac{\pi}{2} k_{vco} A_c A_v \gg (2\pi k_f A_m + 4\pi c)$$

$$\text{i.e.,} \quad k_{vco} A_c A_v \gg (4 k_f A_m + 8c)$$

This condition is required for the case when $k_{vco} A_c A_v \gg 8c$.

Next, we consider the more interesting case when

$$8c \gg k_{vco} A_c A_v.$$

$$\varphi_e(f) = \frac{-k_f M(f) e^{-j2\pi f\tau} jf}{\left((jf)^2 + \left(\frac{k_{vco} A_c A_v C}{2}\right) + j\left(\frac{k_{vco} A_c A_v}{2}\right) f\right)}$$

$$= \frac{-k_f M(f) e^{-j2\pi f\tau} jf}{\left[\left(jf + \frac{k_{vco} A_c A_v}{4}\right)^2 + \left(\frac{k_{vco} A_c A_v C}{2} - \frac{(k_{vco} A_c A_v)^2}{16}\right)\right]}$$

we had earlier considered the case when

$$\frac{k_{vco} A_c A_v C}{2} < \frac{(k_{vco} A_c A_v)^2}{16}$$

i.e., $8C < k_{vco} A_c A_v$.

Now we consider the case when

$k_{vco} A_c A_v < 8C$, in which case

we get

$$\varphi_e(f) = \frac{-k_f M(f) e^{-j2\pi f\tau} jf}{\left[\left(jf + \frac{k_{vco} A_c A_v}{4}\right)^2 - (j\xi)^2\right]}$$

where $\xi = \sqrt{\frac{k_{vco} A_c A_v C}{2} - \frac{(k_{vco} A_c A_v)^2}{16}} > 0$.

$$\therefore \varphi_e(f) = \frac{-k_f M(f) e^{-j2\pi f\tau} jf}{\left[\left(j(f+\xi) + \frac{k_{vco} A_c A_v}{4}\right)\left(j(f-\xi) + \frac{k_{vco} A_c A_v}{4}\right)\right]}$$

using partial fractions we get . . .

$$\frac{-\phi_e(f)}{k_f\, M(f)\, e^{-j\pi f \tau}} = \frac{\left(\frac{1}{2} - j\frac{k_{vco} A_c A_v}{8 z_f}\right)}{\left(jf + \left(jz_f + \frac{k_{vco} A_c A_v}{4}\right)\right)} + \frac{\left(\frac{1}{2} + j\frac{k_{vco} A_c A_v}{8 z_f}\right)}{\left(jf + \left(\frac{k_{vco} A_c A_v}{4} - j z_f\right)\right)}$$

corresponds to an LTI system with impulse response

$$g(t) = \left(2\pi \cos 2\pi z_f t + \frac{\pi k_{vco} A_c A_v}{2 z_f} \sin 2\pi z_f t\right) e^{-\pi \frac{k_{vco} A_c A_v t}{2}} u(t).$$

$$\therefore \phi_e(t) = -k_f \left(m(t-\tau) \circledast g(t)\right)$$

convolution.

Note that $g(t)$ can also be expressed as

$$g(t) = \sqrt{4\pi^2 + \frac{\pi^2 (k_{vco} A_c A_v)^2}{4 z_f^2}} \cos(2\pi z_f t - \delta)\, e^{-\pi \frac{k_{vco} A_c A_v t}{2}} u(t)$$

where $\delta \triangleq \tan^{-1}\left(\frac{k_{vco} A_c A_v}{4 z_f}\right)$.

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We note that $g(t)$ is essentially a damped sinusoid of frequency ζ_p.

~~Since $m(t)$ is a sinusoid of frequency f_m, and~~

~~$\phi_e(t) = -k_f\,(m(t-\tau) \circledast g(t))$, it is~~

~~clear that $\phi_e(t)$ is small if $f_m \ll \zeta_p$~~

~~i.e., $f_m \ll \sqrt{\dfrac{k_{vco}A_cA_vC}{2} - \dfrac{(k_{vco}A_cA_v)^2}{16}}$.~~

Exact expression for $\phi_e(t)$ $(t \geq \tau + T_p)$ is given by

$$\phi_e(t) = -k_f \int_0^{\infty} g(x)\, m(t-\tau-x)\, dx \qquad \Bigg|\; \begin{array}{l} t-\tau-x \gg T_p \\ x < t-\tau-T_p \end{array}$$

$$= -k_f \int_0^{(t-\tau-T_p)} g(x)\, m(t-\tau-x)\, dx$$

$$= -k_f A_m \int_0^{(t-\tau-T_p)} \cos(2\pi\zeta_p x - \delta_p) \cos 2\pi f_m (t-\tau-x-T_p)\, e^{-\pi \frac{k_{vco}A_c}{2} A_v x} \frac{dx}{\sqrt{4\pi^2 + \frac{\pi^2}{4\zeta_p^2}(k_{vco}A_cA_v)^2}}$$

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$\therefore$

$$\frac{-\varphi_e(t)}{2\pi k_f A_m} = \sqrt{1+\left(\frac{k_{vco} A_L A_V}{4 z_1}\right)^2} \int_0^{t-\tau-T_p} \cos(2\pi z_1 x - \delta) \cos 2\pi f_m (t-\tau-x-T_p)\, e^{-\frac{\pi k_{vco} A_L A_V x}{2}}\, dx$$

$$= \frac{\sqrt{1+\left(\frac{k_{vco} A_L A_V}{4 z_1}\right)^2}}{2} \int_0^{(t-\tau-T_p)} \Big\{ \cos\big(2\pi x (z_1 + f_m) - \delta - 2\pi f_m t + 2\pi f_m \tau + 2\pi f_m T_p\big) + \cos\big(2\pi x (z_1 - f_m) - \delta + 2\pi f_m t - 2\pi f_m \tau - 2\pi f_m T_p\big) \Big\}\, e^{-\frac{\pi k_{vco} A_L A_V x}{2}}\, dx$$

from where it can be shown that a sufficient condition which guarantees that $|\varphi_e(t)| \ll 1$ for all $t \geq \tau + T_p$ is

$$\frac{2 k_f A_m \sqrt{1+\left(\frac{k_{vco} A_L A_V}{4 z_1}\right)^2}}{(k_{vco} A_L A_V)} \ll 1.$$

In the regime where $c \gg \frac{k_{vco} A_L A_V}{8}$ this condition reduces to

$$\boxed{k_{vco} A_L A_V \gg 2 k_f A_m}$$

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Comparision between the first and second order PLL's.

PLL HOLD-IN FREQUENCY RANGE:

This refers to the maximum deviation in the frequency of the input signal from the carrier frequency, such that the PLL can still lock (i.e., $\phi_e(t)$ is small).

FIRST ORDER-PLL:

We required that

$$\Delta f_{max} = k_f A_m << \frac{k_{vco} A_c A_v}{2}.$$

SECOND ORDER PLL:

For a second order PLL with $C >> \frac{k_{vco} A_c A_v}{8}$, we required that

$$\Delta f_{max} = k_f A_m << \frac{k_{vco} A_c A_v}{2}.$$

i.e., in both the first & second order PLL's k_{vco} can be used to control the PLL HOLD-IN FREQUENCY RANGE.

LOOP BANDWIDTH:

We define loop bandwidth to ~~the~~ be the maximum possible bandwidth of $m(t)$ such that $v(t)$ is a distortion-free scalar multiple of $m(t-\tau)$.

Consider an input $m(t)$ consisting of two sinusoids

$$m(t) = \begin{cases} A_m \cos 2\pi f_{m_1}(t-T_p) + A_m \cos 2\pi f_{m_2}(t-T_p) & , t \geq T_p \\ 0 & , t < T_p . \end{cases}$$

Then using (16), the output $v(t)$ is given by

$$v(t) \approx A_m \Big\{ G_1 \cos\left(2\pi f_{m_1}(t-\tau-T_p) - \tan^{-1}\frac{2\pi f_{m_1}}{a}\right) + G_2 \cos\left(2\pi f_{m_2}(t-\tau-T_p) - \tan^{-1}\left(\frac{2\pi f_{m_2}}{a}\right)\right)\Big\}$$

Consider an example where

$$f_{m_1} = \frac{a}{2\pi} \text{ and } f_{m_2} = \frac{a}{2\pi}(\sqrt{3})$$

then since $G(f) = \dfrac{a}{\sqrt{a^2+(2\pi f)^2}}$, $G(f_{m_1}) = \dfrac{1}{\sqrt{2}}$ and $G(f_{m_2}) = \frac{1}{2}$

$\therefore$ the output

$$v(t) \simeq \frac{k_f A_m}{2 k_{vco}} \left(\sqrt{2} \cos\left(2\pi f_{m_1}(t-\tau-T_p) - \pi/4\right) + \cos\left(2\pi f_{m_2}(t-\tau-T_p) - \pi/3\right)\right).$$

which is a distorted version of $m(t-\tau)$ due to the different phase shifts i.e., ($-\pi/4$ and $-\pi/3$) in the constituent sinusoids of $m(t)$ (Draw this in Matlab and see the distortion produced)

For the first order PLL loop:

$$G(f) = \frac{a}{\sqrt{a^2+(2\pi f)^2}}, \text{ and } \alpha(f) = \tan^{-1}\left(\frac{2\pi f}{a}\right).$$

$\therefore$ the only way to control distortion in the first order loop is to ensure that $G(f) \simeq 1$ and $\alpha(f) \propto f$ (i.e., linear phase)

Given the expressions $G(f)$ and ~~tan~~ $\alpha(f)$ (above), it is clear that $G(f) \simeq 1$ and $\alpha(f) \propto f$ if and only if $|f| \ll \left(\frac{a}{2\pi}\right)$.

therefore for the first order loop the loop bandwidth (BL) must satisfy

$$BL \ll \frac{a}{2\pi} = \frac{k_{vco} A_c A_V}{2}.$$

$\therefore$ ~~the~~ In the first order PLL the same parameter $k_{vco} A_c A_V$ controls both the loop bandwidth as well as the PLL lock-in frequency range.

For a second order PLL, we have (using equation (25))

$$\frac{V(f)}{M(f)e^{-j2\pi f\tau}} = \frac{n k_f A_c A_V (c + jf)}{(n k_{vco} A_c A_V c + j n k_{vco} A_c A_V f - 2\pi f^2)}$$

from where it is clear that for f such that

$$n\, n k_{vco} A_c A_V c \gg 2\pi f^2$$

i.e. $f \ll \sqrt{\frac{k_{vco} A_c A_V c}{2}}$ we have:

$$\frac{V(f)}{M(f)e^{-j\pi f\tau}} \simeq \frac{n k_f A_c A_V (c + jf)}{(n k_{vco} A_c A_V c + j n k_{vco} A_c A_V f)}$$

$$= \frac{k_f}{k_{vco}} \quad \left(f \ll \sqrt{\frac{k_{vco} A_c A_V}{2} c}\right)$$

In a second order loop if $m(t)$ has its highest frequency component $\ll \sqrt{\frac{k_{VCO} A_c A_V C}{2}}$

then

$V(t) \simeq \frac{k_f}{k_{VCO}} m(t-\tau)$, i.e., the PLL output $V(t)$ will be a <u>distortion-free</u> delayed version of $m(t)$.

DISCUSSION (SUMMARY)

In a first order PLL, both the PLL lock-in frequency range and the loop bandwidth are controlled by the same parameter $\frac{k_{VCO} A_c A_V}{2}$. However the gain of the PLL is $\frac{k_f}{k_{VCO}}$, i.e., a large k_{VCO} will imply a loss in the amplitude at the PLL output. There is therefore a trade-off between a higher loop bandwidth/PLL lock-in frequency range and the PLL gain (in the case of a first order loop). However in the case of a second-order PLL, by choosing $C \gg \frac{k_{VCO} A_c A_V}{8}$, we can control the PLL lock-in frequency range using $\frac{k_{VCO} A_c A_V}{2}$, and independently control the loop bandwidth using C (since loop bandwidth $= O\left(\sqrt{\frac{k_{VCO} A_c A_V C}{2}}\right)$

∴ If we want to have a larger PLL loop bandwidth without increasing $\frac{K_{VCO} A_L A_V}{2}$, then we can do so in a second-order PLL using the control parameter C.

	First-order PLL	Second-order PLL
PLL HOLD-IN FREQUENCY	$\ll \frac{K_{VCO} A_L A_V}{2}$	$\ll \frac{K_{VCO} A_L A_V}{2}$
PLL LOOP BANDWIDTH	$\ll \frac{K_{VCO} A_L A_V}{2}$	$\ll \sqrt{\frac{K_{VCO} A_L A_V C}{2}}$ (assumes $C \gg \frac{K_{VCO} A_L A_V}{8}$)