Lecture 8 EEL 306 (Jan 28). (1)
2014.

Power spectral density of a wide sense stationary random process.

For a w.s.s. $W(t)$, we would like to define a function $S_W(f)$ which describes the distribution of the total power of $W(t)$ in the frequency domain.

If such a function did exist, then the power of $W(t)$ in the frequency band $[f-\frac{\Delta f}{2}, f+\frac{\Delta f}{2}]$ would be

$$\int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} S_W(f)\,df.$$

(In general the power in the freq band $[f_1, f_2]$ is $\int_{f_1}^{f_2} S_W(f)\,df$).

If Δf is small then.

$$\int_{f-\frac{\Delta f}{2}}^{f+\Delta f/2} S_W(f)\,df \approx \Delta f\, S_W(f)$$

$\therefore$ The power of $W(t)$ in the frequency band $[f-\frac{\Delta f}{2}, f+\frac{\Delta f}{2}] \approx \Delta f\, S_W(f)$

or in other words

$$S_W(f) \approx \frac{\text{Power of } W(t) \text{ in } [f-\frac{\Delta f}{2}, f+\frac{\Delta f}{2}]}{\Delta f}$$

The formal definition for $S_W(f)$ then comes out ~~falls~~ naturally as

$$S_W(f) = \lim_{\Delta f \to 0} \frac{\text{Power of } W(t) \text{ in } [f-\frac{\Delta f}{2}, f+\frac{\Delta f}{2}]}{\Delta f}$$

(if the limit Exists)

$S_W(f)$ is aptly called power spectral density because just like the probability density function is the density of the probability distribution, $S_W(f)$ is the density of the power ~~spec~~ in the frequency domain.

In the following we consider the following setup to derive an expression for $S_W(f)$.

Narrow band filter with passband $[f - \frac{\Delta f}{2}, f + \frac{\Delta f}{2}]$.

then based on our previous discussion

$$S_W(f) = \lim_{\Delta f \to 0} \frac{E|N(t)|^2}{\Delta f} \quad \text{--- (1)}$$

$$E[|N(t)|^2] = E[N(t) N^*(t)]$$

since $N(t) = \int h(\tau) W(t-\tau) d\tau$, we have

$$E[|N(t)|^2] = E\left[\int h(\tau_1) W(t-\tau_1) h^*(\tau_2) W^*(t-\tau_2) \, d\tau_1 d\tau_2\right]$$

$$= \iint h(\tau_1) h^*(\tau_2) \underline{E[W(t-\tau_1) W^*(t-\tau_2)]} \, d\tau_1 d\tau_2 \quad \text{--- (2)}$$

this is the autocorrelation between $W(t-\tau_1)$ and $W(t-\tau_2)$.

we define the autocorrelation function

$$R_W(t_1, t_2) \triangleq E[W(t_1) W^*(t_2)].$$

We consider the special case where (4)

$W(t)$ is such that:

$R_W(t_1, t_2)$ depends only on the time difference $(t_1 - t_2)$.

Such processes are called wide sense stationary. (the mean should also be constant) Assuming $W(t)$ to be w.s.s., ~~we get~~ we define the autocorrelation fn.

$$R_W(\tau) \triangleq E\left[W(t)\, W^*(t-\tau)\right]. \longrightarrow (3)$$

$\therefore$ using this in (2) we get:

$$E\left[|N(t)|^2\right] = \iint h(\tau_1)\, h^*(\tau_2)\, R_W(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

with change of integration variables

$$(\tau_1, \tau_2) \longrightarrow (\tau_1, \tau)$$

where $\tau \triangleq \tau_2 - \tau_1$

we get

$$E\left[|N(t)|^2\right] = \iint h(\tau_1)\, h^*(\tau+\tau_1)\, R_W(\tau)\, d\tau_1\, d\tau$$

$$= \int R_W(\tau) \left[\int h(\tau_1)\, h^*(\tau+\tau_1)\, d\tau_1\right] d\tau \longrightarrow (4)$$

We next derive an expression for the inner integral in (4)

To compute (4) we use the Parseval's theorem which states that

$$\int x(t)\, y^*(t)\, dt = \int X(f)\, Y^*(f)\, df. \quad \text{—} (5)$$

Note that $(\tau_1 \to t)$

$$\int h(\tau_1)\, h^*(\tau+\tau_1)\, d\tau_1 = \int h(t)\, h^*(t+\tau)\, dt.$$

with $x(t) \triangleq h(t)$ and

$y(t) \triangleq h(t+\tau)$ we ~~have~~ have

$X(f) = H(f)$ and $Y(f) = H(f)\, e^{j2\pi f\tau}$

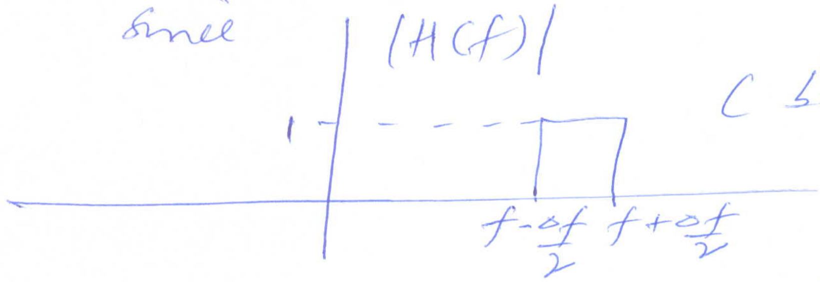
using (5) we get

$\therefore \int h(t)\, h^*(t+\tau)\, dt$

$$= \int H(f)\, H^*(f)\, e^{-j2\pi f\tau}\, df$$

$$= \int |H(f)|^2\, e^{-j2\pi f\tau}\, df$$

Since



Narrow band (bandpass filter centered at f)

$$\therefore \int h(t)\, h^*(t+\tau)\, dt = \int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} e^{-j2\pi f\tau}\, df$$

$$= \Delta f\, \mathrm{sinc}(\Delta f\, \tau)\, e^{j2\pi f\tau} \quad \text{—} (6)$$

using (6) in (4) we get

$$E[|N(t)|^2] = \int R_W(\tau)\, \Delta f\, e^{-j2\pi f\tau}\, \mathrm{sinc}(\Delta f \tau)\, d\tau$$

and hence using (1) we get.

$$S_W(f) = \lim_{\Delta f \to 0} \frac{E|N(t)|^2}{\Delta f}$$

$$= \lim_{\Delta f \to 0} \int R_W(\tau)\, e^{-j2\pi f\tau}\, \mathrm{sinc}(\Delta f \tau)\, d\tau$$

$$= \int R_W(\tau)\, e^{-j2\pi f\tau}\, d\tau .$$

One can also show that

$$R_W(\tau) = \int S_W(f)\, e^{j2\pi f\tau}\, df\,; \quad \text{i.e.,}$$

$$\therefore \quad S_W(f) \rightleftharpoons R_W(\tau)$$

the P.S.D and the autocorrelation function form a Fourier Transform pair.