Lecture 9 EEL 306

Jan 29, 2014

Transmission of a W.S.S. through a ~~random~~ LTI filter.

Let $N(t)$ be a wide sense stationary (W.S.S) random process which is the input to a LTI filter characterized by its impulse response $h(t)$.

$N(t)$ therefore satisfies the following

P-i) $E[N(t_1)] = E[N(t_2)]$ for any t_1, t_2

P-ii) $E[N(t_1) N^*(t_2)] = E[N(t_1+T) N^*(t_2+T)]$ for any t_1, t_2, T.

From P-ii) we can define the autocorrelation function of $N(t)$ as

$$R_N(\tau) = E[N(t) N^*(t-\tau)]. \qquad (1)$$

We will show that $W(t)$ is also W.S.S. (2)
We need to therefore prove that $W(t)$ also satisfies the W.S.S. properties P-i) and P-ii).

We firstly have

$$W(t) = \int h(\tau)\, N(t-\tau)\, d\tau \quad \text{——} \; (2)$$

$$\therefore\; E[W(t)] = E\left[\int h(\tau)\, N(t-\tau)\, d\tau\right]$$

$$= \int h(\tau)\, E[N(t-\tau)]\, d\tau$$ (since $E[\cdot]$ is a linear operator)

$$= \int h(\tau)\, E[N(0)]\, d\tau$$ (since $N(\cdot)$ is W.S.S, mean is constant)

$$= E[N(0)] \int h(\tau)\, d\tau$$

which is a constant and doesnot depend upon t.

This prove P-i) for $W(t)$. Next we try to prove P-ii) for $W(t)$.

$$E[W(t_1) W^*(t_2)] = E\left[\int h(\tau_1) N(t_1-\tau_1)\, d\tau_1 \int h^*(\tau_2) N^*(t_2-\tau_2)\, d\tau_2\right]$$

$$= E\left[\iint h(\tau_1) h^*(\tau_2) N(t_1-\tau_1) N^*(t_2-\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \iint h(\tau_1) h^*(\tau_2)\, E[N(t_1-\tau_1) N^*(t_2-\tau_2)]\, d\tau_1\, d\tau_2 \quad \text{——} \; (3)$$

Since $N(t)$ is w.s.s, from P-ii) we get (3)

$$E[N(t_1-\tau_1)N^*(t_2-\tau_2)] = R_N(\tau_2-\tau_1+t_1-t_2).$$

$\therefore$ using this in (3) we get

$$E[W(t_1)W^*(t_2)]$$

$$= \iint h(\tau_1)h^*(\tau_2)\, R_N(\tau_2-\tau_1+t_1-t_2)\, d\tau_1\, d\tau_2 \quad \text{— (4)}$$

which clearly depends only on the time difference (t_1-t_2).

We will next study the conditions under which the output process $W(t)$ has finite power (i.e., $E|W(t)|^2 < \infty$).

$$E|W(t)|^2 = R_W(0) \quad \text{(using (4) we get)}$$

$$= \iint h(\tau_1)\, h^*(\tau_2)\, R_N(\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

$$\leq \iint |h(\tau_1)\, h^*(\tau_2)\, R_N(\tau_2-\tau_1)|\, d\tau_1\, d\tau_2$$

$$\leq R_N(0) \iint |h(\tau_1)\, h^*(\tau_2)|\, d\tau_1\, d\tau_2$$

where we have used the fact that for any w.s.s $|R_N(\tau)| \leq R_N(0)$.

4

$\therefore E|W(t)|^2 \leq R_N(0) \int\int |h(\tau_1)||h(\tau_2)| d\tau_1 d\tau_2$

$= R_N(0) \left[\int |h(\tau)| d\tau\right]^2.$

From ~~we ca~~ here, we can conclude that $E|W(t)|^2 < \infty$, if $R_N(0) = E|N(t)|^2 < \infty$ and $\int |h(\tau)| d\tau < \infty$, i.e., if the input has finite power and the LTI filter is stable.

Properties of p.s.d.

a) $S_N(f) \geq 0$ for all f.

(from definition, $S_N(f)\,\delta f$ is the power in a narrow band $[f-\frac{\delta f}{2}, f+\frac{\delta f}{2}]$ ~~centered~~ centered at f, and since power is non-negative)

b) If $N(t)$ is real, then

$S_N(f) = S_N(-f).$

Ex: $X(t) = A\cos(2\pi f_c t + \theta)$, $\theta \sim \text{unif}[-\pi, \pi]$.

$S_X(f) = \frac{A^2}{4}\left[\delta(f-f_c) + \delta(f+f_c)\right].$