1) (7 Marks) Prove the Parseval's Theorem which states that for any two complex-valued signals $x(t)$ and $y(t)$

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \;=\; \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

where $X(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$ and $y(t)$ respectively, and * denotes complex conjugation. (**Hint**: Consider a LTI system whose impulse response is $y^*(-t)$ and whose input is $x(t)$. Derive expression for the output $z(t)$ at time $t = 0$ in two different ways, one using the convolution integral and another by taking the inverse Fourier transform of $Z(f)$)

Answer: $x(t) \longrightarrow \boxed{\begin{matrix} y^*(-t) \\ = h(t) \end{matrix}} \longrightarrow z(t)$

$$z(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} y^*(-\tau)\, x(t-\tau)\, d\tau$$

$$\therefore z(0) = \int_{-\infty}^{\infty} y^*(-\tau)\, x(-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \quad \text{——} \; (1)$$

$$Z(f) = X(f)\, H(f) \;;\; H(f) = \text{Fourier}(y^*(-t)) = Y^*(f)$$

$$\therefore Z(f) = X(f)\, Y^*(f) \quad \text{——} \; (2)$$

We know that $Z(f) = \int_{-\infty}^{\infty} z(t)\, e^{-j2\pi f t}\, dt$ and $z(t) = \int_{-\infty}^{\infty} Z(f)\, e^{j2\pi f t}\, df$. Using the latter, we have

$$z(0) = \int_{-\infty}^{\infty} Z(f)\, df = \underbrace{\int_{-\infty}^{\infty} X(f)\, Y^*(f)}_{(3)} \quad (\text{using } (2))$$

Equating (1) and (3) finishes the proof.

2) (6 Marks) Let $x(t)$ be a real-valued band-limited passband signal whose complex envelope is denoted by $\tilde{x}(t)$. Find the value of $\frac{\int_{-\infty}^{\infty} x^2(t)\,dt}{\int_{-\infty}^{\infty} |\tilde{x}(t)|^2\,dt}$. You can assume that $X(f) = 0$, $|f \pm f_c| > W$ and that $f_c > W$. **Hint**: Use the relation $x(t) = \text{Real}\left(\tilde{x}(t)\, e^{j2\pi f_c t}\right)$

Answer: Let $x(t) = Re[\tilde{x}(t) e^{j2\pi f_c t}]$ (using Eulers identity)

$$= \frac{\tilde{x}(t) e^{j2\pi f_c t} + \tilde{x}^*(t) e^{-j2\pi f_c t}}{2}$$

$$\therefore x^2(t) = |x(t)|^2 = x(t)\, x^*(t)$$

$$= \frac{\tilde{x}^2(t) e^{j4\pi f_c t}}{4} + \frac{(\tilde{x}^*(t))^2 e^{-j4\pi f_c t}}{4} + \frac{|\tilde{x}(t)|^2}{2}.$$

$$\therefore \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2}\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 dt + \frac{1}{4}\int_{-\infty}^{\infty} \tilde{x}^2(t) e^{j4\pi f_c t}\,dt + \frac{1}{4}\int_{-\infty}^{\infty} (\tilde{x}^*(t))^2 e^{-j4\pi f_c t}\,dt$$

Let $y(t) \triangleq \tilde{x}^2(t)$.

Since $X(f) = 0$ for $|f \pm f_c| > W$,

it follows that $\tilde{x}(t)$ is bandlimited to $[-W, W]$

and therefore $y(t)$ is band limited to $[-2W, 2W]$ (i.e; $Y(f) = 0$ for $|f| > 2W$)

$$\therefore \int_{-\infty}^{\infty} \tilde{x}^2(t) e^{j4\pi f_c t}\,dt = \int_{-\infty}^{\infty} y(t) e^{j4\pi f_c t}\,dt = Y(f = -2f_c)$$

$= 0$ since $f_c > W$ and $Y(f) = 0$ for $|f| > 2W$.

$$\therefore \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2}\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 dt$$, and ratio is therefore $1/2$.

3) (7 Marks) Find the complex envelope of the passband signal

$$x(t) = A\,(1 + k\,m(t))\,\cos(2\pi f_c t + \phi)$$

where $A > 0, k \in \mathbb{R}, \phi \in (-\pi, \pi]$ are constants. $m(t)$ is a real-valued baseband signal band-limited to $[-W, W]$. Assume $f_c > W$.

Answer:

$$x(t) = A\,(1 + k m(t)) \cos(2\pi f_c t + \phi)$$

$$= \text{Real}\left[A\,(1 + k m(t))\, e^{j(2\pi f_c t + \phi)}\right]$$

(since $A, k, m(t)$ are real)

$$= \text{Real}\left[A\,(1 + k m(t))\, e^{j\phi}\, e^{j2\pi f_c t}\right] \quad \text{—} (1)$$

$$= \text{Real}\left[\tilde{x}(t)\, e^{j2\pi f_c t}\right] \quad \text{—} (2)$$

From the uniqueness of the complex baseband representation, it follows by comparing equations (1) and (2) that

$$\tilde{x}(t) = A\,(1 + k m(t))\, e^{j\phi}.$$

4) (7 Marks) Consider a channel whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x(t) - \frac{1}{2}x(t - t_0)$$

where t_0 is a constant. Assuming the real-valued input $x(t)$ to be a band-limited passband signal (i.e., $X(f) = 0$, $|f \pm f_c| > W$), find the expression for a base band signal $h_b(t)$ (band-limited to $[-W, W]$) such that

$$\tilde{y}(t) = \int_{-\infty}^{\infty} h_b(\tau)\, \tilde{x}(t - \tau)\, d\tau.$$

Here, $\tilde{y}(t)$ and $\tilde{x}(t)$ are the complex envelopes of $y(t)$ and $x(t)$ respectively.

Answer: The impulse response of the channel is given by $h(t) = \delta(t) - \frac{1}{2}\delta(t-t_0)$.
Note that $h(t)$ is not bandlimited.
Since the input $x(t)$ is anyways band-limited to $[f_c-W, f_c+W]$ and $[-f_c-W, -f_c+W]$, an equivalent pass band channel (which is band limited to the pass band of $x(t)$) is obtained by convoluting $h(t)$ with the passband filter $z(t)$ having ~~characteristics~~ Fourier transform

$$Z(f) = \begin{cases} 1, & |f \pm f_c| < W \\ 0, & \text{otherwise.} \end{cases}$$

$\therefore z(t) = 4W \operatorname{sinc}(2Wt) \cos 2\pi f_c t.$

$\therefore h_{eq}(t) = \underset{\text{(convolution)}}{h(t) * z(t)} = 4W \operatorname{sinc}(2Wt)\cos 2\pi f_c t - 2W \operatorname{sinc}(2W(t-t_0)) \cos 2\pi f_c (t-t_0)$

~~Further~~ $= \operatorname{Re}\left[\tilde{h}_{eq}(t) e^{j2\pi f_c t}\right]$

where $\tilde{h}_{eq}(t) = 4W \operatorname{sinc}(2Wt) - 2W \operatorname{sinc}(2W(t-t_0)) e^{-j2\pi f_c t_0}$

Since $\tilde{y}(t) = \frac{1}{2}\int \tilde{h}_{eq}(\tau)\, \tilde{x}(t-\tau)\, d\tau$, it follows that

FINAL ANSWER $\Longrightarrow$ $h_b(t) = 2W \operatorname{sinc}(2Wt) - W \operatorname{sinc}(2W(t-t_0)) e^{-j2\pi f_c t_0}.$

5) (3 Marks) Let a real-valued wide sense stationary random process $X(t)$ be such that its autocorrelation function is periodic, i.e.,

$$\begin{aligned} R_X(\tau) &\triangleq \mathbb{E}[X(t)X(t-\tau)] \\ &= R_X(\tau+T) \end{aligned}$$

for some finite $T > 0$.

Prove that the random process $X(t)$ is also periodic with the same period T, i.e., $X(t) = X(t+T)$ with probability one (i.e., for some realization $x_\omega(t)$ of the random process ~~$X(t)$~~ it might happen that $x_\omega(t) \neq x_\omega(t+T)$, but then the probability measure of all such realizations is 0.

Answer: Let $Z(t) \triangleq X(t) - X(t+T)$ be a random process.

$$E(Z^2(t)) = E\left[(X(t) - X(t+T))^2\right]$$

$$= E[X^2(t)] + E(X^2(t+T))$$

$$-2E(X(t)X(t+T))$$

$$= R_X(0) + R_X(0) - 2R_X(T)$$

$$= 2(R_X(0) - R_X(T)).$$

However, since $R_X(\tau) = R_X(\tau+T)$, we have $R_X(0) = R_X(T)$, and therefore

$$E(Z^2(t)) = E[(X(t) - X(t+T))]^2 = 0,$$

this then implies that

$X(t) = X(t+T)$ (with probability one)