SOLUTIONS TO

MINOR - II EXAM

E.E.L. 308.

Dr. SAIF K. MOHAMMED,

E.E., I.I.T. DELHI

Q.1.

$$N(t) \rightarrow \boxed{h(t)} \rightarrow X(t)$$

linear filter

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Since $N(t)$ is strictly stationary, it follows that $X(t)$ is also strictly stationary.

therefore $E[X(t)X(t-\tau)]$ will only depend on τ and will be independent of t.

We also know that the power spectral densities of $N(t)$ and $X(t)$ are related by

$$S_X(f) = S_N(f)\,|H(f)|^2$$

$$= \frac{N_0}{2}|H(f)|^2 \quad \text{since } S_N(f) = \frac{N_0}{2}$$

(i.e., $N(t)$ is white)

Also, since the autocorrelation function and the power spectral density form a Fourier transform pair, we have

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 e^{j2\pi f\tau}\, df$$

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this can also be written in another way. (2)

Since $H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt$

we have $H^*(f) = \int_{-\infty}^{\infty} h^*(t) e^{j2\pi ft} dt$

$\therefore |H(f)|^2 = H(f) H^*(f)$

$$= \left(\int_{-\infty}^{\infty} h(t_1) e^{-j2\pi ft_1} dt_1\right)\left(\int_{-\infty}^{+\infty} h^*(t_2) e^{+j2\pi ft_2} dt_2\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1) h^*(t_2) e^{-j2\pi ft_1} e^{j2\pi ft_2} dt_1 dt_2 . \quad \text{—— (2)}$$

using (2) in (1) we get.

$$R_X(\tau) = \frac{N_0}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1) h^*(t_2) e^{-j2\pi ft_1} e^{j2\pi ft_2} e^{j2\pi f\tau} df\, dt_1\, dt_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1) h^*(t_2) \left[\int_{-\infty}^{\infty} e^{j2\pi f(\tau + t_2 - t_1)} df\right] dt_1 dt_2 \quad \text{—— (3)}$$

(~~Here τ is the dummy variable and (t₁, t₂) are given.~~)

we know that

$$\delta(t - t_0) \rightleftharpoons e^{-j2\pi ft_0} \quad \text{—— (5)}$$

using (4) we have

$$\int_{-\infty}^{\infty} e^{-j2\pi ft_0} e^{j2\pi ft} df$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f(t - t_0)} df = \delta(t - t_0) \quad \text{—— (5)}$$

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using ⑤ we therefore have

$$\int_{-\infty}^{\infty} e^{j\pi f(\tau-(t_1-t_2))}\, df$$

$$= \delta(\tau-(t_1-t_2)). \quad — ⑥$$

using ⑥ in ③ we get

$$R_X(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h^*(t_2)\ \delta(\tau-t_1+t_2)\, dt_1 dt_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(t_1)\ \delta(\tau+t_2-t_1)\, dt_1\right] h^*(t_2)\, dt_2 \quad — ⑦$$

we know that for any function $x(t)$

$$\int x(t)\,\delta(\tau-t)\, dt = x(\tau) \quad — ⑧$$

$$\therefore \int h(t_1)\ \delta((\tau+t_2)-t_1)\, dt_1$$

$$= h(\tau+t_2). \quad — ⑨$$

using ⑨ in ⑦ we finally get

$$R_X(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h^*(t_2)\, h(\tau+t_2)\, dt_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h(t_2)\, h(\tau+t_2)\, dt_2 \quad \text{(since } h(t) \text{ is real)}$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h(t)\, h(t-\tau)\, dt \quad \text{(change of variable } t_2 \to t\text{)}$$

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b) Since $N(t)$ is a strictly stationary gaussian random process, $X(t)$ is also a strictly stationary gaussian random process. $\therefore$ For any $t_1 \neq t_2$, $X(t_1)$ and $X(t_2)$ are jointly gaussian random variables.

Using ③ we get

$$E[X(t_1)X(t_2)] = \cancel{\int h(t)\, h(t+(t_2-t_1))}\, dt$$
$$= \int h(t)\, h(t+(t_1-t_2))\, dt.$$

From Parseval's theorem

$$\int x(t)\, y^*(t)\, dt = \int X(f)\, Y^*(f)\, df.$$

choosing $x(t) = h(t)$ and $y(t) = h(t+(t_1-t_2))$

we have

$$\int h(t)\, h^*(t+t_1-t_2)\, dt = \int h(t)\, h(t+t_1-t_2)\, dt$$

(since $h(t)$ is real)

$$= \int H(f)\, H^*(f)\, e^{-j\pi f(t_1-t_2)}\, df$$

For $h(t) = 2B\, \text{sinc}(2Bt)$,

$$H(f) = \begin{cases} 1, & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

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$\therefore \int h(t)\, h^*(t+t_1-t_2)\,dt = \int_{-\infty}^{\infty} |H(f)|^2 e^{-j2\pi f(t_1-t_2)}\,df$

$= \int_{-B}^{B} e^{-j2\pi f(t_1-t_2)}\,df$

$= 2B\ \mathrm{sinc}(2B(t_2-t_1))$

$\therefore E[X(t_1)\,X(t_2)] = 2B\ \mathrm{sinc}(2B(t_2-t_1))$.

Since $X(t_1)$ and $X(t_2)$ have zero mean and are jointly Gaussian, they are statistically independent if and only if

$E\left[\left[X(t_1)-E(X(t_1))\right]\left[X(t_2)-E(X(t_2))\right]\right]=0$

which is true if and only if

$\mathrm{sinc}\,2B(t_2-t_1)=0$

$\Rightarrow |t_2-t_1| = \frac{n}{2B}$ for some integer $n \neq 0$.

Dr. SAIF
(E.E., I.I.T. DELHI)

Q.2.

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Since $X(t)$ is stationary

$E[X(t)]$ is a constant (w.r.t time)

$E[X(t)] = 2$ for all t

Therefore

$$E[X(t_k)] = 2. \quad \text{——— (1)}$$

$$E[X^2(t_k)] = R_X(0)$$

$$= \int_{-\infty}^{\infty} S_X(f)\,df \quad \text{——— (2)}$$

Variance of $X(t_k)$ is given by

$$E\left[\left(X(t_k) - E[X(t_k)]\right)^2\right]$$

$$= E[X^2(t_k)] - \left(E[X(t_k)]\right)^2$$

$$= \int_{-\infty}^{\infty} S_X(f)\,df - 4 \quad (\text{using } (2))$$

Since variance is always positive we have $\int_{-\infty}^{\infty} S_X(f)\,df - 4 \geq 0$.

Since $X(t)$ is a Gaussian random process, $X(t_k)$ is Gaussian distributed with mean 2 and variance $\sigma_X^2 \triangleq \left(\int_{-\infty}^{\infty} S_X(f)\,df - 4\right)$. Its pdf is therefore

$$f_{X(t_k)}(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\, e^{-(x-2)^2/2\sigma_X^2}$$

where $\sigma_X^2 \triangleq \int_{-\infty}^{\infty} S_X(f)\,df - 4$.

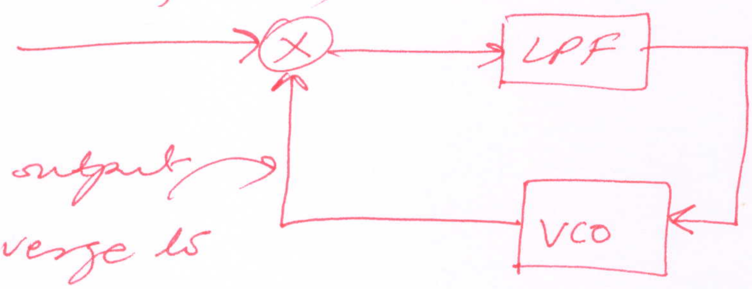
a) <u>During the synchronization phase,</u> the unmodulated carrier

$s(t) = \sin 2\pi f_c t$ is only transmitted.

The receiver receives

$$r(t) = s(t-\tau) = \sin 2\pi f_c (t-\tau)$$

$r(t) = \sin 2\pi f_c (t-\tau)$



the VCO output will converge to

$$\sin\left[2\pi f_c (t-\tau) + \frac{\pi}{2}\right]$$

i.e., the VCO output will have a phase which leads the phase of $r(t)$ by $+90^\circ$.

$\therefore$ At the end of the synchronization phase the VCO o/p is $\sin\left[2\pi f_c (t-\tau) + \pi/2\right]$

$$= \cos\left[2\pi f_c (t-\tau)\right].$$

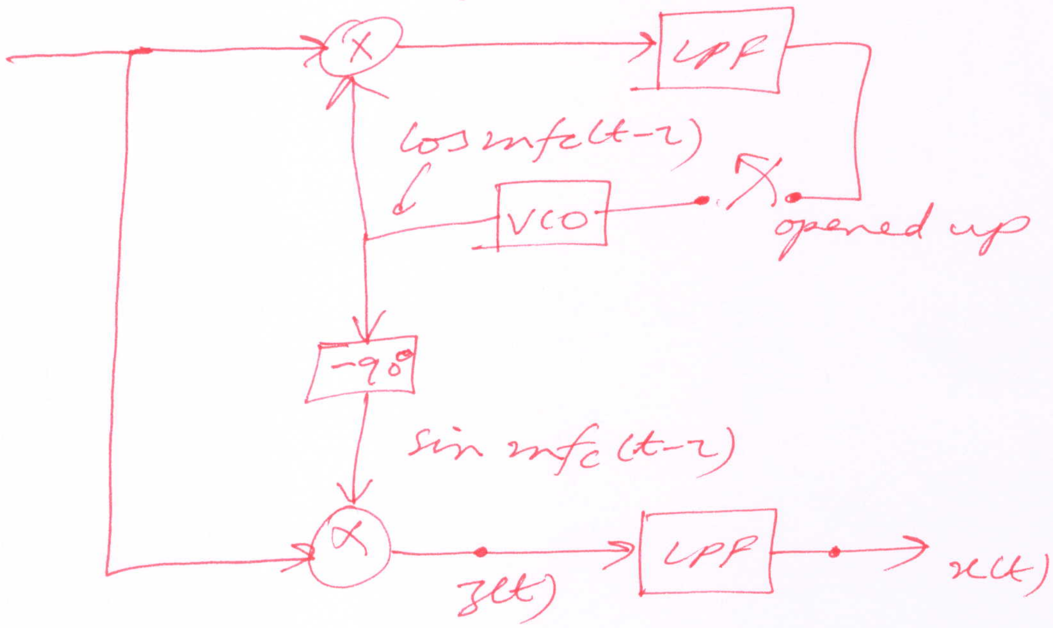
DR. SAIF
I.I.T. DELHI

b) COHERENT DEMODULATOR. ~~8~~ 8

During message phase the PLL loop is opened up. ~~and the demodulator~~

$r(t) = s(t-\tau)$

$= m(t-\tau) \sin 2\pi f_c(t-\tau)$



c) the signal $z(t)$ is given by

$$z(t) = r(t) \sin 2\pi f_c(t-\tau)$$

$$= m(t-\tau) \sin^2 2\pi f_c(t-\tau)$$

$$= \frac{m(t-\tau)}{2}\left[1 - \cos 4\pi f_c(t-\tau)\right].$$

$$= \frac{m(t-\tau)}{2} - \frac{m(t-\tau)}{2}\cos 4\pi f_c(t-\tau) \quad (1)$$

(* we have taken $W \triangleq B$)

~~has~~ ~~otherwise~~ is band limited to $[-W, W]$, which is exactly the pass band of the LPF.

the L.P.F rejects this term since it has power in the stopband of the low pass filter (LPF)

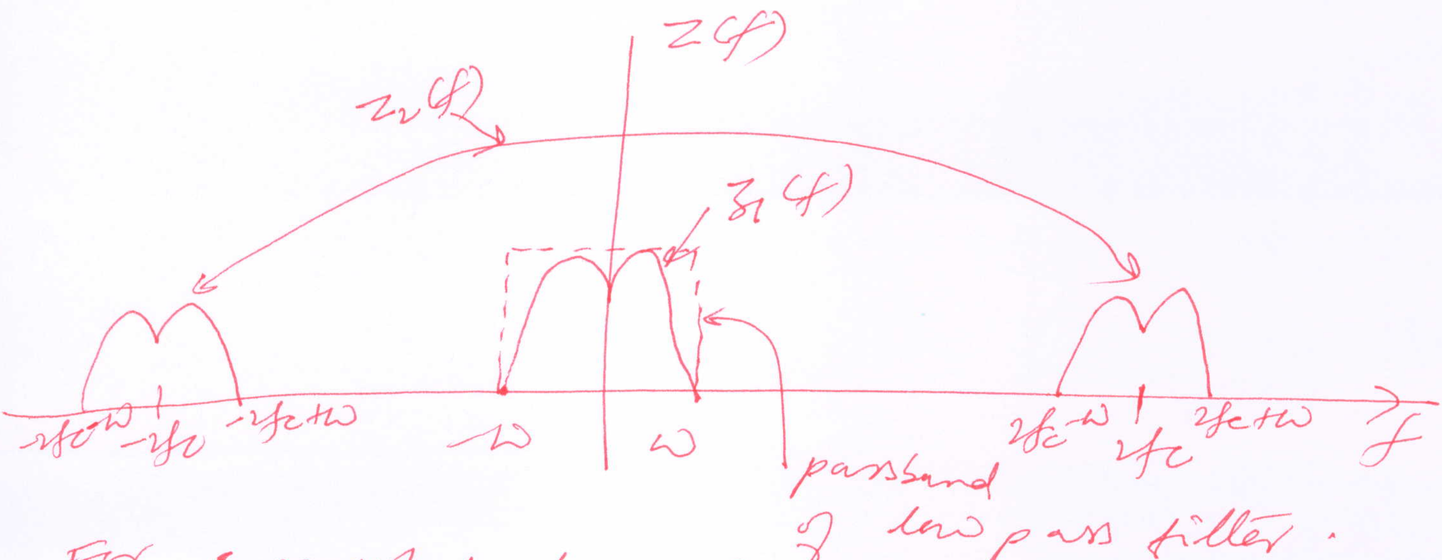
$\therefore$ LPF output $x(t) = \frac{m(t-\tau)}{2}$.

d) In ① in part c, let

$$z_1(t) \triangleq \frac{m(t-\tau)}{2} \text{ and}$$

$$z_2(t) \triangleq \frac{m(t-\tau)}{2} \cos 4\pi f_c (t-\tau)$$

Note that $Z_1(f)$ is restricted to $[-W, W]$, whereas $Z_2(f)$ is restricted to $(2f_c - W, 2f_c + W)$ and $(-2f_c - W, -2f_c + W)$ as illustrated below.



For successful demodulation $z_2(t)$ must be rejected by the filter. This is possible if and only if the intervals $[-W, W]$, $[2f_c - W, 2f_c + W]$ and $[-2f_c - W, -2f_c + W]$ are disjoint.

For $[-W, W]$, $[2f_c-W, 2f_c+W]$, $[-2f_c-W, -2f_c+W]$ to be disjoint intervals, it is necessary and sufficient that

$$2f_c - W > W$$

i.e., $f_c > W$

Carrier frequency must be larger than the highest frequency component in $m(t)$.

DR. SAIF,
E.E., I.I.T. DELHI

Q.4

$x(t) = m(t)\, c(t)$.

Fourier series of $c(t)$

Since $c(t) = c(t+T_p)$ is periodic it has a Fourier series representation i.e.,

$$c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_p t + \sum_{n=1}^{\infty} b_n \sin 2\pi n f_p t.$$

where $f_p = \frac{1}{T_p}$ is the fundamental frequency.

$$b_n = \frac{2}{T_p}\int_0^{T_p} c(t) \sin 2\pi n f_p t \, dt$$

$$= \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} c(t) \sin 2\pi n f_p t \, dt$$

$$= \frac{2}{T_p}\int_{-T_p/2}^{0} -\sin 2\pi n f_p t \, dt + \frac{2}{T_p}\int_0^{T_p/2} \sin 2\pi n f_p t \, dt$$

$$= \frac{4}{T_p}\int_0^{T_p/2} \sin 2\pi n f_p t \, dt = \frac{4\left[-\cos 2\pi n f_p t\right]_0^{\frac{T_p}{2}}}{T_p \cdot (2\pi n f_p)}$$

$$= \frac{2}{\pi n}\left\{1 - \cos \frac{2\pi n}{2}\right\}.$$

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$$\therefore b_n = \frac{2}{\pi n}\{1 - (-1)^n\}$$

$$a_0 = \frac{1}{T_p}\int_0^{T_p} c(t)\,dt = 0.$$

$$a_n = \frac{2}{T_p}\int_0^{T_p} c(t)\cos 2\pi n f_p t\,dt$$

$$= 0.$$

$$\therefore c(t) = 2\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\pi n}\sin 2\pi n f_p t.$$

$$\therefore x(t) = m(t)\,c(t)$$

$$= 2\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\pi n}\, m(t)\sin 2\pi n f_p t.$$

The n-th term $m(t)\sin 2\pi n f_p t$ has energy/power in the frequency band $[nf_p - W, nf_p + W]$. Assuming $f_p > 2W$, the bands of ~~each~~ all terms can be seen to be non-overlapping. Note that the term corresponding to $n=1$ is $\frac{4}{\pi}m(t)\sin 2\pi f_p t$, which is a DSB-SC signal. This signal can be obtained by passing $x(t)$ through an ideal band pass filter with pass band $[f_p - W, f_p + W]$. $\therefore$ $s(t) = \frac{4}{\pi}m(t)\sin 2\pi f_p t.$