Solution to Major Exam EEL-306 (1)

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1.

a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2\pi f_m (t-\tau)}{\tau}\, d\tau$.

$$= \int_{-\infty}^{\infty} \frac{(\sin 2\pi f_m t \cos 2\pi f_m \tau - \cos 2\pi f_m t \sin 2\pi f_m \tau)}{\tau} d\tau$$

$$= \sin 2\pi f_m t \int_{-\infty}^{\infty} \frac{\cos 2\pi f_m \tau}{\tau} d\tau - \cos 2\pi f_m t \int_{-\infty}^{\infty} \frac{\sin 2\pi f_m \tau}{\tau} d\tau$$

$\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2\pi f_m \tau}{\tau} d\tau = 0$ since $\dfrac{\cos 2\pi f_m \tau}{\tau}$ is an odd function of τ

$$\therefore \int_{-\infty}^{\infty} \frac{\sin 2\pi f_m (t-\tau) d\tau}{\tau} = -\cos 2\pi f_m t \int_{-\infty}^{\infty} \frac{\sin 2\pi f_m \tau}{\tau} d\tau$$

$$= -\cos 2\pi f_m t \int_{-\infty}^{\infty} \frac{\sin \pi x}{x} dx$$

$$= -\pi \cos 2\pi f_m t \int_{-\infty}^{\infty} \mathrm{sinc}\, x\, dx \quad \text{substitution } x = 2 f_m \tau$$

$$= -\pi \cos 2\pi f_m t.$$

b) $m(t) \sin 2\pi f_c t = \mathrm{Re}\left(-j m(t)\, e^{j 2\pi f_c t}\right)$

$\therefore$ the complex envelope of $m(t)\sin 2\pi f_c t$ is $-j m(t)$.

c) $x(t)\sin 2\pi f_c t \longrightarrow$ ENVELOPE DETECTOR $\longrightarrow |x(t)|$

the output of the envelope detector is $|x(t)|$.

d) SSB transmits only one sideband since the other sideband contains the same information.
On the other hand both the sidebands are transmitted in DSB-SC.
Therefore SSB needs only half as much bandwidth (as DSB-SC) to send the same information.

e) Vestigial sideband modulation is used when the message has sufficient power at low (near zero) frequency, due to which SSB cannot be used since it requires a filter to select one of the sidebands (and this filter in practice will have a nonzero pass band to stop band transition gap which will distort the message signal if the message has significant power in low frequency components).

Q.2.

We have $v(t) = \frac{-A_c A_v}{2} \sin \phi_e(t)$

where $\phi_e(t)$ is the phase difference between the vco o/p and the input (received) FM signal.

From the lectures we also know that for $t \geq 0$.

$$\frac{-d\phi_e(t)}{dt} = a \sin \phi_e(t) + b \quad , \text{ where}$$

$a = \pi k_{vco} A_c A_v$ and $b = 2\pi k_f A_m$

since $m(t) = A_m$, and the synchronization phase ends at $t=0$ (ensuring that $\phi_e(t)=0$) we have

$$\phi_e(t=0) = 0.$$

Since it is given that $\phi_e(t)$ is small we make the light approximation, $\sin \phi_e(t) \simeq \phi_e(t)$. With this approximation, the differential equation becomes

$$\frac{-d\phi_e(t)}{dt} = a\phi_e(t) + b \quad , \quad \phi_e(t=0) = 0$$

$$\therefore \int_{\phi_e(t=0)}^{\phi_e(t)} \frac{d\phi_e(t)}{a\phi_e(t)+b} = -\int_0^t dt$$

$$\therefore \left[\ln(a\phi_e(t)+b)\right]_{\phi_e(t=0)}^{\phi_e(t)} = -at$$

$$\therefore \ln\left(1 + \frac{a\phi_e(t)}{b}\right) = -at \quad \text{since } \phi_e(t=0) = 0$$

$$\therefore \phi_e(t) \simeq -b/a \left(1 - e^{-at}\right)$$

Since $\phi_e(t) \simeq -b/a\,(1-e^{-at})$ decreases monotonically starting at $\phi_e(t=0)=0$ at $t=0$ and converging to $\phi_e(t) \simeq -b/a$ as $t\to\infty$.

Since $v(t) = -\frac{A_c A_v}{2} \sin\phi_e(t)$

$\therefore$ $v(t) \simeq -\frac{A_c A_v}{2}\phi_e(t)$ (small $\phi_e(t)$)

$$= -\frac{A_c A_v}{2} \cdot -b/a\,(1-e^{-at})$$

$$= \frac{A_c A_v}{2}\frac{b}{a}(1-e^{-at})$$

Note that $v(t)$ increases monotonically with increasing t, and ~~therefore there~~

$$\lim_{t\to\infty} v(t) = \frac{A_c A_v}{2}\cdot\frac{b}{a}$$

$$= \frac{A_c A_v}{2}\cdot\frac{2\pi k_f A_m}{\pi k_{vco} A_c A_v}$$

$$= \frac{k_f A_m}{k_{vco}}.$$

Since $0.9\frac{k_f}{k_{vco}}A_m < \frac{k_f A_m}{k_{vco}}$, and $v(t)$ is monotonically increasing in t, there exists a $t=t_c$ such that

$v(t_c) = 0.9\frac{k_f}{k_{vco}}A_m$ and for all

$t > t_c$, $v(t) > \frac{0.9 k_f A_m}{k_{vco}}$

this critical time $t = t_c$ is therefore given by (3)

$v(t_c)$

$$= \frac{A_c A_v}{2} \, b/a \, (1 - e^{-at})$$

$$= \frac{A_c A_v}{2} \cdot \frac{2 n k_f A_m}{n k_{vco} A_c A_v} \left(1 - e^{-n k_{vco} A_c A_v t}\right)$$

$$= \frac{k_f A_m}{k_{vco}} \left(1 - e^{-n k_{vco} A_c A_v t_c}\right)$$

$$= \frac{0.9 k_f}{k_{vco}} A_m$$

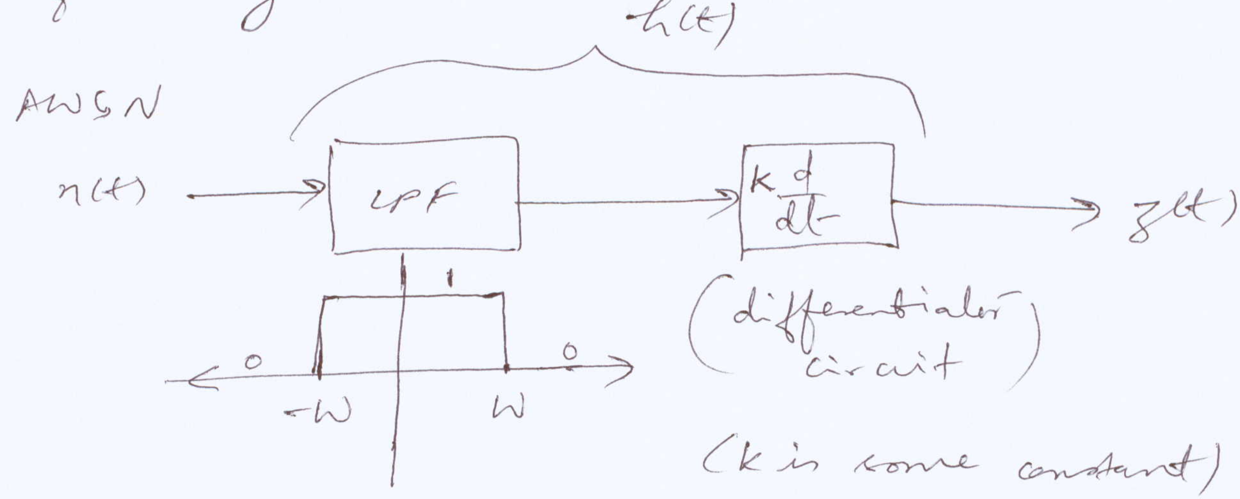
$$\therefore \quad e^{-n k_{vco} A_c A_v t_c} = 0.1$$

or $\quad \ln 10 = n k_{vco} A_c A_v t_c$

$$\Rightarrow \quad t_c = \frac{\ln 10}{n k_{vco} A_c A_v}.$$

We need to find $E[z(t)^2]$ in the following. ①

3.



(k is some constant).

The differentiator circuit in combination with the LPF acts as a low pass filter have frequency response.

$$H(f) = \begin{cases} j2\pi f k, & |f| < W \\ 0, & \text{otherwise}. \end{cases}$$

∴ Since $n(t)$ is a stationary zero mean AWSN, $z(t)$ is also a stationary zero mean Gaussian random process, having power spectral density.

$$S_Z(f) = S_N(f)\,|H(f)|^2$$

$$= \begin{cases} k^2 (2\pi f)^2 \frac{N_0}{2}, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

②

$$\therefore \quad E(z^2(t)) = \int_{-\infty}^{+\infty} S_z(f)\, df$$

$$= \int_{-W}^{W} \frac{N_0}{2} (2\pi f)^2 k^2 \, df$$

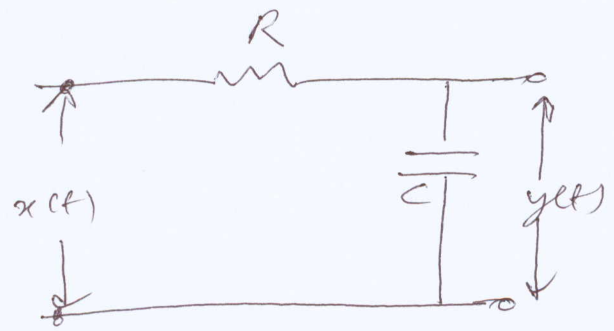
$$= 2 N_0 \pi^2 \times \frac{2W^3}{3} k^2$$

$$= \frac{4\pi^2 k^2}{3} N_0 W^3 .$$

①

~~6.1~~

4.



$x(t) = A_c \cos 2\pi f_c t + w(t)$.

let $y(t)$ denote the output of this system.

the filter response is given by

$$H(f) = \frac{1/j2\pi fC}{R + \frac{1}{j2\pi fC}} = \frac{1}{1 + j2\pi fRC}.$$

$$\therefore \quad |H(f)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}}$$

In the steady state

$$\therefore \quad y(t) = \frac{A_c}{\sqrt{1 + (2\pi f_c RC)^2}} \cos(2\pi f_c t + \phi(f_c)) + w(t) * h(t)$$

convolution

where $h(t)$ = Inverse Fourier $(H(f))$ is the impulse response of the filter.

$\phi(f)$ = phase response of the filter.

let $w_h(t) \triangleq w(t) \otimes h(t)$.

Since $w(t)$ is zero mean stationary AWGN, it follows that $w_h(t)$ is also a zero mean stationary ~~AWGN~~ Gaussian noise (random process).

The power spectral density of w_h is given by

$$S_{w_h}(f) = \frac{N_0}{2} |H(f)|^2$$

$$= \frac{N_0}{2} \cdot \frac{1}{(1+(2\pi fRC)^2)}.$$

$\therefore$ Signal power at filter output $= \dfrac{A_c^2}{2(1+(2\pi f_c RC)^2)}$, and

noise power at filter output $= \displaystyle\int_{-\infty}^{\infty} S_{w_h}(f)\,df$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{df}{(1+(2\pi fRC)^2)}.$$

$$= \frac{N_0}{2}\cdot\frac{1}{(2\pi RC)}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)} \qquad x \triangleq 2\pi fRC$$

$$= \frac{N_0}{4\pi RC}\left[\tan^{-1}x\right]_{-\infty}^{\infty}$$

$$= (N_0/4RC)$$

(3)

$\therefore$ The SNR at the output of the filter $= \dfrac{A_c^2}{2(1+(2\pi f_c RC)^2)} \times \dfrac{4RC}{N_0}$

$$= \frac{2A_c^2 \, RC}{N_0 \, (1+(2\pi f_c RC)^2)}$$

(1)

5. $r(t) = A_I \cos 2\pi(f_c + 2W)t + A_I \cos 2\pi(f_c + \frac{7W}{2})t$

$y(t) = a\, r(t) + b\, r^3(t).$

the band pass filter (BPF) rejects signals outside the band $[f_c - W, f_c + W]$,

$\therefore BPF(y(t)) = a\, BPF(r(t)) + b\, BPF(r^3(t))$
$= b\, BPF(r^3(t))$

because $BPF(r(t)) = 0$ since $r(t)$ contains no signal in the band $[f_c - W, f_c + W]$

$$r^3(t) = A_I^3 \left(\cos 2\pi(f_c + 2W)t + \cos 2\pi(f_c + \tfrac{7W}{2})t \right)^3$$

$$= A_I^3 \cos^3 2\pi(f_c + 2W)t + A_I^3 \cos^3 2\pi(f_c + \tfrac{7W}{2})t$$
$$+ 3A_I^3 \cos^2 2\pi(f_c + 2W)t \cos 2\pi(f_c + \tfrac{7W}{2})t$$
$$+ 3A_I^3 \cos 2\pi(f_c + 2W)t \cos^2 2\pi(f_c + \tfrac{7W}{2})t$$

$\therefore$

$$BPF(r^3(t)) = 3A_I^3\, BPF\left\{ \cos 2\pi(f_c + \tfrac{7W}{2})t \cos^2 2\pi(f_c + 2W)t \right\}$$
$$+ 3A_I^3\, BPF\left\{ \cos 2\pi(f_c + 2W)t \cos^2 2\pi(f_c + \tfrac{7W}{2})t \right\}$$

since

$$\cos^3 2\pi(f_c + 2W)t = \frac{3 \cos 2\pi(f_c + 2W)t}{4} + \frac{\cos 2\pi(3f_c + 6W)t}{4}$$

contains no signal in the band $[f_c - W, f_c + W]$.

$\therefore BPF\{\cos^3 2\pi(f_c + 2W)t\} = 0$

Similarly $BPF\{\cos^3 2\pi(f_c+\frac{7W}{2})t\} = 0.$ ②

Further:

$$BPF(r^3(t)) = \frac{3A_I^3}{2} BPF\{(1+\cos 2\pi(2f_c+4W)t) \cos 2\pi(f_c+\frac{7W}{2})t\}$$

$$+ \frac{3A_I^3}{2} BPF\{(1+\cos 2\pi(2f_c+7W)t) \cos 2\pi(f_c+2W)t\}$$

$$= \frac{3}{2}A_I^3 BPF\{\cos 2\pi(f_c+\frac{7W}{2})t\}^{\nearrow 0}$$

$$+ \frac{3A_I^3}{4} BPF\{2\cos 2\pi(f_c+\frac{7W}{2})t \cos 2\pi(2f_c+4W)t\}$$

$$+ \frac{3A_I^3}{2} BPF\{\cos 2\pi(f_c+2W)t\}^{\nearrow 0}$$

$$+ \frac{3A_I^3}{4} BPF\{2\cos 2\pi(f_c+2W)t \cos 2\pi(2f_c+7W)t\}$$

$$= \frac{3A_I^3}{4} BPF\{\cos 2\pi(3f_c+\frac{15W}{2})t + \cos 2\pi(f_c+\frac{W}{2})t\}$$

$$+ \frac{3A_I^3}{4} BPF\{\cos 2\pi(3f_c+9W)t + \cos 2\pi(f_c+5W)t\}$$

$$= \frac{3A_I^3}{4} \cos 2\pi(f_c+\frac{W}{2})t.$$

$\therefore BPF\{y(t)\} = \frac{3bA_I^3}{4} \cos 2\pi(f_c+\frac{W}{2})t$ which falls inside the communication band $[f_c-W, f_c+W]$.

(1)

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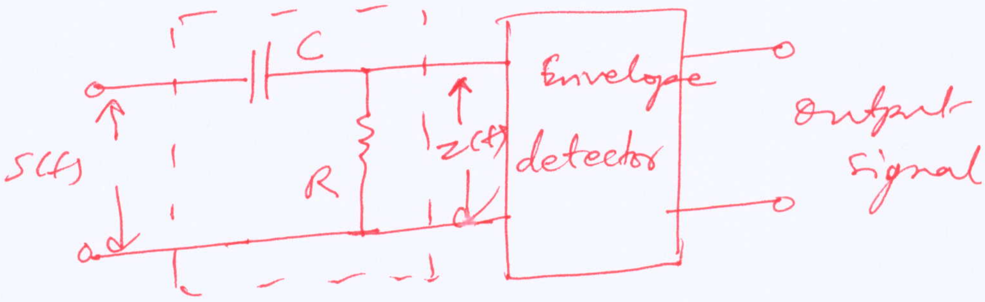
~~6-16~~

6.
$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(x)\,dx\right] \quad \text{— (1)}$$



It is given that for all significant frequency components of s(t).

$$R \ll \frac{1}{j2\pi f C} \quad \text{— (2)}$$

the response of the high pass filter ⇒

is given by

$$H(f) = \frac{R}{R + \frac{1}{j2\pi f C}} = \frac{j2\pi f RC}{1 + R\cdot j2\pi f C}$$

(3)

$\simeq j2\pi f RC$ for all f where s(t) has significant components. (follows from (2))

Note that

$\therefore$ $H(f) \cong j2\pi f RC$, acts like

a differentiating system with a gain

factor of RC.

We have

$$Z(f) = \text{Fourier}(z(t))$$

$$= S(f)\,H(f).$$

$$= \left(S(f)\,j2\pi f\right)RC$$

$$\therefore z(t) = RC\,\frac{ds(t)}{dt}$$

$$= RC\,\frac{d}{dt}\left(A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right)\right)$$

$$= -RC\,A_c \sin\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right)\left(2\pi f_c + 2\pi k_f m(t)\right)$$

$$z(t) = -2\pi f_c RC A_c \sin\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right)\left(1 + \frac{k_f m(t)}{f_c}\right).$$

The output of the envelope detector is therefore

$$V(t) = \left| 2\pi f_c R C A_c \left(1 + \frac{k_f m(t)}{f_c} \right) \right|$$

$$= 2\pi f_c R C A_c \left(1 + \frac{k_f m(t)}{f_c} \right)$$

Since it is given that

$k_f |m(t)| < f_c$ for all t.

this shows us how an envelope detector can also be used for FM demodulation.