EEL-306

Solution to Tutorial-3 problems.

Prob. 1 soln:

let

$$Z^I(t) \triangleq N(t)\cos(2\pi f_c t + \theta) \text{ and}$$

$$Z^Q(t) \triangleq -N(t)\sin(2\pi f_c t + \theta),$$

then it is clear that

$$N_0^I(t) = Z^I(t) \circledast h(t) \text{ , and}$$

convolution

$$N_0^Q(t) = Z^Q(t) \circledast h(t), \text{ where}$$

$h(t)$ is the impulse response of the ideal low pass filter having Fourier transform

$$H(f) = \begin{cases} 1, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

From our lectures we know that

$$S_{Z^I}(f) = S_{Z^Q}(f) = \frac{S_N(f-f_c) + S_N(f+f_c)}{4}$$

since $S_N(f) = \frac{N_0}{2}$ for all f ($N(t)$ is white)

we have

$$S_{Z^I}(f) = S_{Z^Q}(f) = \frac{N_0}{4}.$$

Further

$$S_{N^I}(f) = |H(f)|^2 S_{Z^I}(f)$$

$$= \begin{cases} \frac{N_0}{4}, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

$$= S_{N^Q}(f).$$

$\therefore R_{N^I}(\tau) = R_{N^Q}(\tau) =$ inverse Fourier of $S_{N^I}(f)$

$$= \frac{N_0 W}{2} \operatorname{sinc}(2W\tau).$$

Note that the average power of

$N^I(t) = E(N^I(t)^2) = E(N^Q(t)^2)$

$$= R_{N^I}(\tau=0) = \frac{N_0 W}{2}.$$

We also note that

$R_{N^I}\left(\tau = \frac{m}{2W}\right) = 0$, i.e., if $N^I(t)$ is

sampled at $2W$ samples per second then the ~~same~~ noise samples are uncorrelated.

Next we compute the cross correlation

$$R_{N^I, N^Q}(\tau) = E\left[N^I(t)\, N^Q(t-\tau)\right]$$

$$= E\left[\int Z^I(t-\tau_1)\, h(\tau_1)\, d\tau_1 \int Z^Q(t-\tau-\tau_2)\, h(\tau_2)\, d\tau_2\right]$$

$$= \iint h(\tau_1) h(\tau_2) \, E\left[Z^I(t-\tau_1) \, Z^Q(t-\tau-\tau_2) \right] d\tau_1 d\tau_2$$

To compute this integral we need to first compute the cross correlation between $Z^I(t)$ and $Z^Q(t)$

$$R_{Z^I,Z^Q}(\tau) \triangleq E\left[Z^I(t) Z^Q(t-\tau) \right]$$

$$= E\left[-N(t) \cos(2\pi f_c t + \theta) \, N(t-\tau) \sin(2\pi f_c (t-\tau) + \theta) \right]$$

$$= -\frac{R_N(\tau)}{2} E\left[2 \sin(2\pi f_c(t-\tau)+\theta) \cos(2\pi f_c t + \theta) \right]$$

$$= \frac{R_N(\tau)}{2} \sin(2\pi f_c \tau)$$

$$= \frac{N_0}{4} \delta(\tau) \sin(2\pi f_c \tau)$$

using this expression above we get

$$R_{N^I,N^Q}(\tau) = \iint h(\tau_1) h(\tau_2) R_{Z^I,Z^Q}(\tau_2 - \tau_1 - \tau) \, d\tau_1 d\tau_2$$

$$= \iint h(\tau_1) h(\tau_2) \frac{N_0}{4} \delta(\tau_2 - \tau_1 - \tau) \sin(2\pi f_c (\tau_2 - \tau_1 - \tau)) \, d\tau_1 d\tau_2$$

$$\cancel{= \frac{N_0 \sin(2\pi f_c \tau)}{4} \int h(\tau_1) h(\tau_1 + \tau) \, d\tau_1}$$

$$= \frac{N_0}{4} \iint h(\tau_1) h(\tau_2) \, \delta(\tau_2 - \tau_1 - \tau) \sin(2\pi f_c (\tau_2 - \tau_1 - \ldots$$

$$= \cancel{\frac{N_0}{4} \iint h(\tau_1)\, h(\tau_1 + \tau_2)}$$

$$= \frac{N_0}{4} \int h(\tau_1)\, h(\tau_1 + \tau)\, \sin(0) \cdot d\tau_1$$

$$= 0 .$$

This shows that $N^I(t)$ and $N^Q(t)$ are uncorrelated.

c) $\tilde{N}(t) = N^I(t) + j N^Q(t)$.

$$R_{\tilde{N}}(\tau) = E[\tilde{N}(t)\, \tilde{N}^*(t-\tau)]$$

$$= E[(N^I(t) + jN^Q(t))\,(N^I(t-\tau) - j N^Q(t-\tau))]$$

$$= E[N^I(t)\, N^I(t-\tau)] + E[N^Q(t)\, N^Q(t-\tau)]$$

$$= R_{N^I}(\tau) + R_{N^Q}(\tau)$$

$$= N_0 W \operatorname{sinc}(2W\tau)$$

$\therefore$ $S_{\tilde{N}}(f) =$ Inverse Fourier transform of $R_{\tilde{N}}(\tau)$

$$= \begin{cases} \frac{N_0}{2}, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

For further reading, refer to section 5.11 in the book titled "Narrowband Noise".

solution to problem 2:

we have $Z \triangleq \int_{-\infty}^{\infty} W(t)\, h(t)\, dt$ (5)

$$E(Z) = \int_{-\infty}^{\infty} E(W(t))\, h(t)\, dt$$

$$= \int_{-\infty}^{\infty} E(W(0))\, h(t)\, dt$$ since $W(t)$ is w.s.s. it has a constant mean

$$= E(W(0))\, H(0)$$

$$E(Z^2) = E\left[\int_{-\infty}^{\infty} W(t_1)\, h(t_1)\, W(t_2)\, h(t_2)\, dt_1\, dt_2\right]$$

$$= \int_{-\infty}^{\infty} E\left(W(t_1)\, W(t_2)\right) h(t_1)\, h(t_2)\, dt_1 dt_2$$

since $W(t)$ is w.s.s.

$$= \int_{-\infty}^{\infty} R_W(t_1 - t_2)\, h(t_1)\, h(t_2)\, dt_1\, dt_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} \delta(t_1 - t_2)\, h(t_1)\, h(t_2)\, dt_1\, dt_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(t_1)^2\, dt_1$$ since $W(t)$ is white

For $E(Z^2) < \infty$, it is clear that $\int_{-\infty}^{\infty} h^2(t_1)\, dt_1 < \infty$.

Solution to problem 3. $Y(t) = X_1(t) + X_2(t)$.

$$E[Y(t)] = E(X_1(t)) + E[X_2(t)]$$

$$= E[X_1(0)] + E[X_2(0)]$$

since $X_1(t)$ and $X_2(t)$ are w.s.s.

$\therefore E[Y(t)]$ does not depend on t.

$$E[Y(t_1)\, Y^*(t_2)] = E\left[[X_1(t_1) + X_2(t_1)][X_1^*(t_2) + X_2^*(t_2)]\right]$$

$$= E[X_1(t_1)\, X_1^*(t_2)] + E[X_2(t_1)\, X_2^*(t_2)]$$

$$+ E[X_1(t_1)\, X_2^*(t_2)] + E[X_2(t_1)\, X_1^*(t_2)]$$

with $t_1 = t$ and $t_2 = t - \tau$, we get

$$E[Y(t)\, Y^*(t-\tau)] = R_{X_1}(\tau) + R_{X_2}(\tau)$$

$$+ E[X_1(t)\, X_2^*(t-\tau)]$$

$$+ E[X_1^*(t-\tau)\, X_2(t)]$$

For $Y(t)$ to be w.s.s,

$E[X_1(t)\, X_2^*(t-\tau)] + E[X_1^*(t-\tau)\, X_2(t)]$ should depend only on τ.

Let

$$R_{X_1,X_2}(t_1,t_2) \triangleq E[X_1(t_1)X_2^*(t_2)],$$ ⑦

then

$$E[Y(t)Y^*(t-\tau)] = R_{X_1}(\tau) + R_{X_2}(\tau) + R_{X_1,X_2}(t, t-\tau) + R^*_{X_1,X_2}(t-\tau, t).$$

So, if $R_{X_1,X_2}(t_1,t_2)$ i.e the cross correlation between $X_1(t)$ and $X_2(t)$ depends only on the time difference t_1-t_2, then it follows that

$E[Y(t)Y^*(t-\tau)]$ depends only on the time difference τ and not on t, i.e. $Y(t)$ is w.s.s.

solution to Prob 5.12 in book:

$X(t)$ and $Y(t)$ are jointly wide sense stationary.

a) What this means is that the cross correlation function

$$R_{XY}(t_1, t_2) = E[X(t_1) Y^*(t_2)]$$

~~also~~ depends only on the time difference $t_1 - t_2$.

$\therefore$ let $R_{XY}(\tau) \triangleq E[X(t) Y^*(t-\tau)]$

then $R_{YX}(-\tau) = E[Y(t) X^*(t+\tau)]$

$$= E[Y(t-\tau) X^*(t)]$$

since the cross correlation does not depend on t.

$$= \left(E[X(t) Y^*(t-\tau)]\right)^*$$

$$= R_{XY}^*(\tau)$$

when $X(t)$ and $Y(t)$ are real-valued processes then it follows that

$$R_{XY}^*(\tau) = R_{YX}(-\tau)$$

b) Consider for a fixed τ,

$$Z(t) \triangleq X(t) - Y(t-\tau),$$

Since $X(t)$ and $Y(t)$ are jointly w.s.s.,
it follows that

$$E(Z^2(t)) = E\left[\left[X(t) - Y(t-\tau)\right]^2\right]$$

is a constant and is clearly non-negative since $[X(t) - Y(t-\tau)]^2$ is non-negative.

$$\therefore E\left[\left(X(t) - Y(t-\tau)\right)^2\right] \geq 0$$

$$\Rightarrow E(X^2(t)) + E(Y^2(t-\tau)) - 2E(X(t)Y(t-\tau)) \geq 0$$

$$\Rightarrow R_X(0) + R_Y(0) \geq 2R_{XY}(\tau)$$

The result now follows.

Solution to Prob. 5.18 in book.

$$X(t) = A \cos(2\pi f t - \theta)$$

where both f and θ are random.

This happens in oscillator circuits which are not accurate enough so that the frequency f of the sine wave generated is not precisely controlled.

The autocorrelation function of $X(t)$ is given by

$$E(X(t)X(t-\tau)) = A^2 E[\cos(2\pi f t - \theta) \cos(2\pi f(t-\tau) - \theta)]$$

$$= \frac{A^2}{2} E[\cos 2\pi f \tau + \cos(2\pi f(2t-\tau) - 2\theta)].$$

Both f and θ are r.v's, and are independent of each other.

$$E[\cos(2\pi f(2t-\tau) - 2\theta)] = \cancel{\iint g(f)\, g(}$$

$$= \iint p_f(f)\, p_\theta(\theta) \cos(2\pi f(2t-\tau - 2\theta)) \, df\, d\theta$$

Here, $P_f(\cdot)$ and $P_\theta(\cdot)$ are the probability density function of f and θ respectively, with

$$P_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \le \theta \le \pi. \end{cases}$$

$$E\left[\cos(2\pi f(2t-\tau) - 2\theta)\right]$$

$$= \int_{-\infty}^{\infty} P_f(f) \left[\int_{-\pi}^{\pi} P_\theta(\theta) \cos(2\pi f(2t-\tau) - 2\theta)\right] d\theta \, df$$

The inner integral

$$\int_{-\pi}^{\pi} P_\theta(\theta) \cos(2\pi f(2t-\tau) - 2\theta) \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos[2\pi f(2t-\tau) - 2\theta] \, d\theta$$

$$= 0$$

$$\therefore E[X(t)X(t-\tau)] = \frac{A^2}{2} E[\cos 2\pi f\tau].$$

$$= \frac{A^2}{2} \int_{-\infty}^{+\infty} \cos 2\pi f\tau \, P_f(f) \, df$$

which depends only on τ and is independent of t.

$\therefore$ $X(t)$ is w.s.s.

we have

$$R_X(\tau) = \frac{A^2}{2} \int_{-\infty}^{\infty} p_f(f) \cos 2\pi f \tau \, d\tau$$

$$= \frac{A^2}{4} \int_{-\infty}^{\infty} p_f(f) \left(e^{j2\pi f \tau} + e^{-j2\pi f \tau} \right) d\tau$$

$$= \frac{A^2}{4} \int_{-\infty}^{\infty} p_f(f) e^{j2\pi f \tau} \, d\tau$$

$$\underbrace{+ \frac{A^2}{4} \int_{-\infty}^{\infty} p_f(f) e^{-j2\pi f \tau} \, d\tau}$$

$$= \frac{A^2}{4} \int_{-\infty}^{\infty} p_f(-f) e^{j2\pi f \tau} \, d\tau.$$

$$\therefore R_X(\tau) = \frac{A^2}{4} \int_{-\infty}^{\infty} \left(p_f(f) + p_f(-f) \right) e^{j2\pi f \tau} \, d\tau.$$

= Fourier transform of

$$\frac{A^2}{4} \left(p_f(f) + p_f(-f) \right)$$

$$\therefore S_X(f) = \frac{A^2}{4} \left[p_f(f) + p_f(-f) \right].$$

where $p_f(f)$ is the probability density function of the oscillator frequency f.

When $f = f_c$ = constant, then $p_f(f) = \dfrac{\delta(f-f_c) + \delta(f+f_c)}{2}$

$\therefore S_X(f) = \dfrac{A^2}{2} \left[\delta(f-f_c) + \delta(f+f_c) \right]$ in this special case.