9. The noisiness of a receiver may also be measured in terms of the so-called *noise figure*. The relationship between the noise figure and the equivalent noise temperature may be found in Haykin and Moher (2005).
10. Discussion of both analytical and statistical techniques of characterizing propagation may be found in Parsons (1992).
11. The statistical characterization of communication systems presented in this book is confined to the first two moments, mean and autocorrelation function (equivalently, autocovariance function) of the pertinent random process. However, when a random process is transmitted through a nonlinear system, valuable information is contained in higher-order moments of the resulting output process. The parameters used to characterize higher-order moments in the time domain are called *cumulants*; their multidimensional Fourier transforms are called *polyspectra*. For a discussion of higher-order cumulants and polyspectra and their estimation, see the papers by Brillinger (1965) and Nikias and Raghuveer (1987).

PROBLEMS

5.1

(a) Show that the characteristic function of a Gaussian random variable X of mean μ_X and variance σ_X^2 is

$$\phi_X(v) = \exp\left(jv\mu_X - \tfrac{1}{2}v^2\sigma_X^2\right)$$

(b) Using the result of part (a), show that the nth central moment of this Gaussian random variable is

$$E[(X-\mu_X)^n] = \begin{cases} 1\times 3\times 5\ldots(n-1)\sigma_X^n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

5.2 A Gaussian-distributed random variable X of zero mean and variance σ_X^2 is transformed by a piecewise-linear rectifier characterized by the input–output relation (see Figure P5.1):

$$Y = \begin{cases} X, & X \geq 0 \\ 0, & X < 0. \end{cases}$$

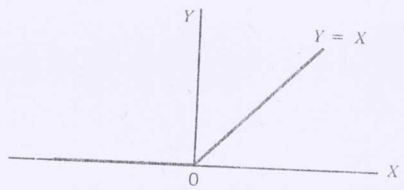

Figure P5.2

The probability density function of the new random variable Y is described by

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ k\delta(y), & y = 0 \\ \dfrac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\dfrac{y^2}{2\sigma_X^2}\right), & y > 0 \end{cases}$$

(a) Explain the physical reasons for the functional form of this result.

(b) Determine the value of the constant k by which the delta function $\delta(y)$ is weighted.

5.3 A binary signal having the value of ± 1 is detected in the presence of additive white Gaussian noise of zero mean and variance σ^2. What is the probability density function of the signal observed at the input to the detector? Derive an expression for the probability that the observed signal is larger than a specified threshold α.

5.4 Consider a random process $X(t)$ defined by

$$X(t) = \sin(2\pi f t)$$

in which the frequency f is a random variable uniformly distributed over the interval $(0, W)$. Show that $X(t)$ is nonstationary. *Hint*: Examine specific sample functions of the random process $X(t)$ for the frequency $f = W/4$, $W/2$, and W, say.

5.5 For a complex random process $Z(t)$, define the autocorrelation function as

$$R_Z(\tau) = \mathbf{E}[Z^*(t)Z(t+\tau)]$$

where $*$ represents complex conjugation. Derive the properties of this complex autocorrelation corresponding to Eqs. (5.64), (5.65), and (5.67).

5.6 For the complex random process $Z(t) = Z_I(t) + jZ_Q(t)$ where $Z_I(t)$ and $Z_Q(t)$ are real-valued random processes given by

$$Z_I(t) = A\cos(2\pi f_1 t + \theta_1)$$

and

$$Z_Q(t) = A\cos(2\pi f_2 t + \theta_2)$$

where θ_1 and θ_2 are uniformly distributed over $[-\pi, \pi]$. What is the autocorrelation of $Z(t)$? Suppose $f_1 = f_2$? Suppose $\theta_1 = \theta_2 = \theta$?

5.7 Let X and Y be statistically independent Gaussian-distributed random variables, each with zero mean and unit variance. Define the Gaussian process

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

(a) Determine the joint probability density function of the random variables $Z(t_1)$ and $Z(t_2)$ obtained by observing $Z(t)$ at times t_1 and t_2, respectively.
(b) Is the process $Z(t)$ stationary? Why?

5.8 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(a) If $X(t)$ contains a dc component equal to A, then $R_X(\tau)$ will contain a constant component equal to A^2.
(b) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ will also contain a sinusoidal component of the same frequency.

5.9 The square wave $x(t)$ of Figure P5.9 of constant amplitude A, period T_0, and delay t_d, represents the sample function of a random process $X(t)$. The delay is random, described by the probability density function

$$f_{T_d}(t_d) = \begin{cases} \dfrac{1}{T_0}, & -\dfrac{1}{2}T_0 \le t_d \le \dfrac{1}{2}T_0 \\ 0, & \text{otherwise} \end{cases}$$

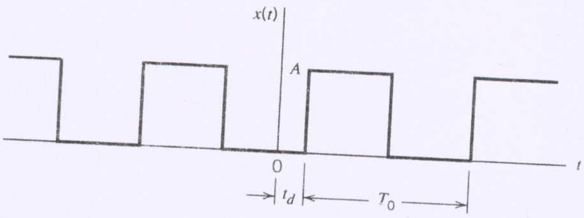

Figure P5.9

(a) Determine the probability density function of the random variable $X(t_k)$ obtained by observing the random process $X(t)$ at time t_k.
(b) Determine the mean and autocorrelation function of $X(t)$ using ensemble averaging.
(c) Determine the mean and autocorrelation function of $X(t)$ using time-averaging.
(d) Establish whether or not $X(t)$ is wide-sense stationary. In what sense is it ergodic?

5.10 A binary wave consists of a random sequence of symbols 1 and 0, similar to that described in Example 5.8, with one basic difference: symbol 1 is now represented by a pulse of amplitude A volts and symbol 0 is represented by zero volts. All other parameters are the same as before. Show that for this new random binary wave $X(t)$:

(a) The autocorrelation function is

$$R_X(\tau) = \begin{cases} \dfrac{A^2}{4} + \dfrac{A^2}{4}\left(1 - \dfrac{|\tau|}{T}\right), & |\tau| < T \\ \dfrac{A^2}{4}, & |\tau| \ge T \end{cases}$$

(b) The power spectral density is

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2 T}{4}\operatorname{sinc}^2(fT)$$

What is the percentage power contained in the dc component of the binary wave?

5.11 A random process $Y(t)$ consists of a dc component of $\sqrt{3/2}$ volts, a periodic component $g(t)$, and a random component $X(t)$. The autocorrelation function of $Y(t)$ is shown in Figure P5.11.

(a) What is the average power of the periodic component $g(t)$?
(b) What is the average power of the random component $X(t)$?

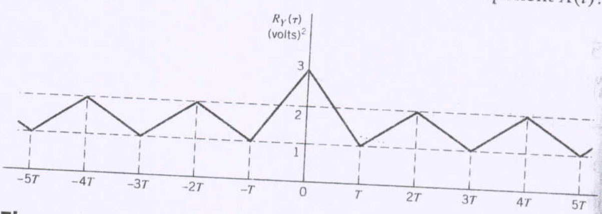

Figure P5.11

5.12 Consider a pair of wide-sense stationary random processes $X(t)$ and $Y(t)$. Show that the cross-correlations $R_{XY}(\tau)$ and $R_{YX}(\tau)$ of these processes have the following properties:

(a) $R_{XY}(\tau) = R_{YX}(-\tau)$
(b) $|R_{XY}(\tau)| \le \frac{1}{2}[R_X(0) + R_Y(0)]$

5.13 Consider two linear filters connected in cascade as in Figure P5.13. Let $X(t)$ be a wide-sense stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.

(a) Find the autocorrelation function of $Y(t)$.
(b) Find the cross-correlation function $R_{VY}(\tau)$ of $V(t)$ and $Y(t)$.

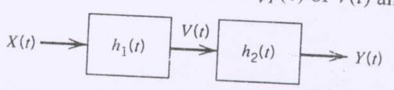

Figure P5.13

5.14 A wide-sense stationary random process $X(t)$ is applied to a linear time invariant filter of impulse response $h(t)$, producing an output $Y(t)$.

(a) Show that the cross-correlation function $R_{YX}(\tau)$ of the output $Y(t)$ and the input $X(t)$ is equal to the impulse response $h(\tau)$

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5.16 A pair of r
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convolved with the autocorrelation function $R_X(\tau)$ of the input, as shown by

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau - u)\, du$$

Show that the second cross-correlation function $R_{XY}(\tau)$ equals

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-u) R_X(\tau - u)\, du$$

(b) Find the cross-spectral densities $S_{YX}(f)$ and $S_{XY}(f)$.

(c) Assuming that $X(t)$ is a white noise process with zero mean and power spectral density $N_0/2$, show that

$$R_{YX}(\tau) = \frac{N_0}{2} h(\tau)$$

Comment on the practical significance of this result.

5.15 The power spectral density of a random process $X(t)$ is shown in Figure P5.15.

(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.

(b) What is the dc power contained in $X(t)$?

(c) What is the ac power contained in $X(t)$?

(d) What sampling rates will give uncorrelated samples of $X(t)$? Are the samples statistically independent?

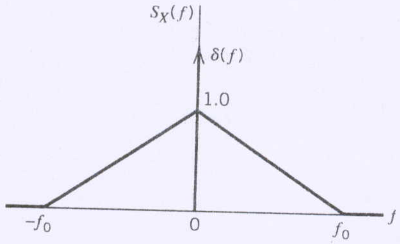

Figure P5.15

5.16 A pair of noise processes $n_1(t)$ and $n_2(t)$ are related by

$$n_2(t) = n_1(t)\cos(2\pi f_c t + \theta) - n_1(t)\sin(2\pi f_c t + \theta)$$

where f_c is a constant, and θ is the value of a random variable Θ whose probability density function is defined by (assume that θ is independent of $n_1(t)$)

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The noise process $n_1(t)$ is stationary and its power spectral density is as shown in Figure P5.16. Find and plot the corresponding power spectral density of $n_2(t)$.

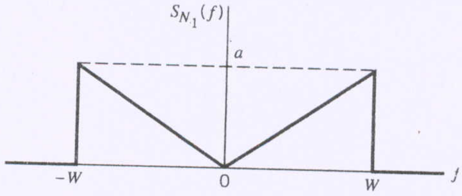

Figure P5.16

5.17 A *random telegraph signal* $X(t)$, characterized by the autocorrelation function

$$R_X(\tau) = \exp(-2\upsilon|\tau|)$$

where υ is a constant, is applied to the low-pass RC filter of Figure P5.17. Determine the power spectral density and autocorrelation function of the random process at the filter output.

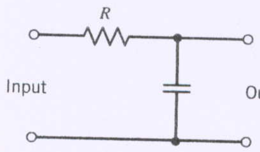

Figure P5.17

5.18 The output of an oscillator is described by

$$X(t) = A\cos(2\pi f t - \Theta),$$

where A is a constant, and f and Θ are independent random variables, The probability density function of Θ is defined by

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Find the power spectral density of $X(t)$ in terms of the probability density function of the frequency f. What happens to this power spectral density when the frequency f assumes a constant value?

5.19 A stationary, Gaussian process $X(t)$ has zero mean and power spectral density $S_X(f)$. Determine the probability density function of a random variable obtained by observing the process $X(t)$ at some time t_k.

5.20 A Gaussian process $X(t)$ of zero mean and variance σ_X^2 is passed through a full-wave rectifier, which is described by the input–output relation of Figure P5.20. Show that the probability density function of the random variable $Y(t_k)$, obtained by observing the random process $Y(t)$ at the rectifier output at time t_k, is as follows.

$$f_{Y(t_k)}(y) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \dfrac{1}{\sigma_X} \exp\left(-\dfrac{y^2}{2\sigma_X^2}\right), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

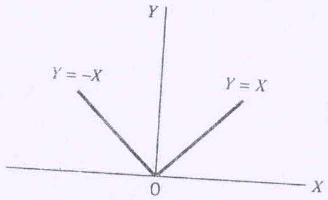

Figure P5.20

5.21 Let $X(t)$ be a zero mean, stationary, Gaussian process with autocorrelation function $R_X(\tau)$. This process is applied to a square-law device defined by the input-output relation

$$Y(t) = X^2(t)$$

where $Y(t)$ is the output.

(a) Show that the mean of $Y(t)$ is $R_X(0)$.
(b) Show that the autocovariance function of $Y(t)$ is $2R_X^2(\tau)$.

5.22 A stationary, Gaussian process $X(t)$ with mean μ_X and variance σ_X^2 is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, as shown in Figure P5.22.

(a) Determine the joint probability density function of the random variables $Y(t_1)$ and $Z(t_2)$.
(b) What conditions are necessary and sufficient to ensure that $Y(t_1)$ and $Z(t_2)$ are statistically independent?

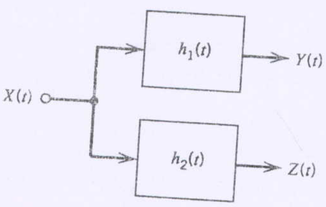

Figure P5.22

5.23 A stationary, Gaussian process $X(t)$ with zero mean and power spectral density $S_X(f)$ is applied to a linear filter whose impulse response $h(t)$ is shown in Figure P5.23. A sample Y is taken of the random process at the filter output at time T.

(a) Determine the mean and variance of Y.
(b) What is the probability density function of Y?

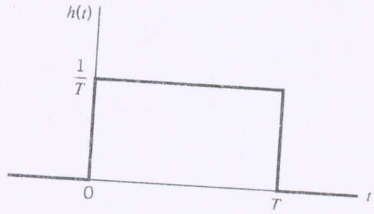

Figure P5.23

5.24 Consider a white Gaussian noise process of zero mean and power spectral density $N_0/2$ that is applied to the input of the high-pass RL filter shown in Figure P5.24.

(a) Find the autocorrelation function and power spectral density of the random process at the output of the filter.
(b) What are the mean and variance of this output?

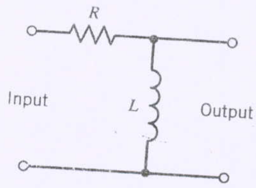

Figure P5.24

5.25 A white noise $w(t)$ of power spectral density $N_0/2$ is applied to a *Butterworth* low-pass filter of order n, whose amplitude response is defined by

$$|H(f)| = \frac{1}{[1 + (f/f_0)^{2n}]^{1/2}}$$

(a) Determine the noise equivalent bandwidth for this low-pass filter.
(b) What is the limiting value of the noise equivalent bandwidth as n approaches infinity?

5.26 The shot-noise process $X(t)$ defined by Eq. (5.114) is stationary. Why?

5.27 White Gaussian noise of zero mean and power spectral density $N_0/2$ is applied to the filtering scheme shown in Figure P5.27. The noise at the low-pass filter output is denoted by $n(t)$. (assume $f_c \gg B$)

(a) Find the power spectral density and the autocorrelation function of $n(t)$.
(b) Find the mean and variance of $n(t)$.
(c) What is the rate at which $n(t)$ can be sampled so that the resulting samples are essentially uncorrelated?

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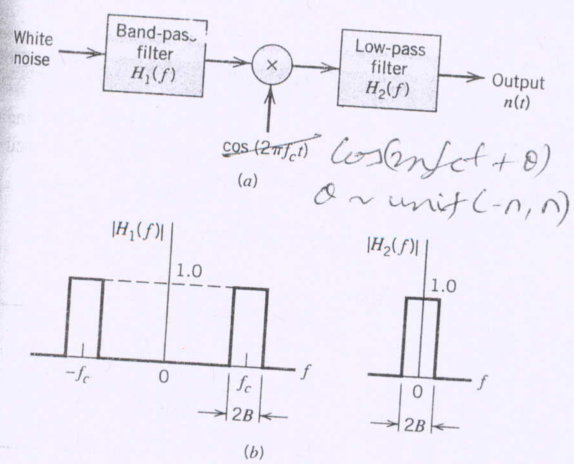

Figure P5.27

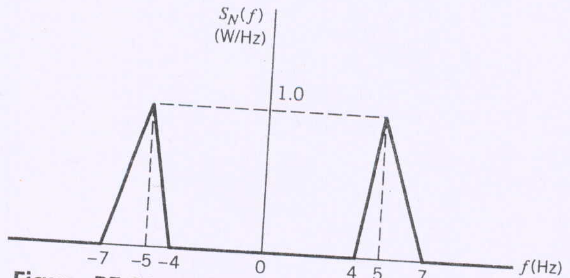

Figure P5.29

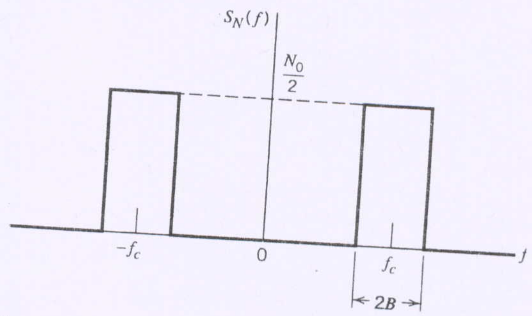

Figure P5.30

5.28 Let $X(t)$ be a stationary process with zero mean, autocorrelation function $R_X(\tau)$, and power spectral density $S_X(f)$. We are required to find a linear filter with impulse response $h(t)$, such that the filter output is $X(t)$ when the input is white noise of power spectral density $N_0/2$.

(a) Determine the condition that the impulse response $h(t)$ must satisfy in order to achieve this requirement.
(b) What is the corresponding condition on the transfer function $H(f)$ of the filter?
(c) Using the Paley–Wiener criterion (see Section 2.7), find the requirement on $S_X(f)$ for the filter to be causal.

5.29 The power spectral density of a narrowband noise $n(t)$ is as shown in Figure P5.29. The carrier frequency is 5 Hz.

(a) Find the power spectral densities of the in-phase and quadrature components of $n(t)$.
(b) Find their cross-spectral densities.

5.30 Consider a Gaussian noise $n(t)$ with zero mean and the power spectral density $S_N(f)$ shown in Figure P5.30.

(a) Find the probability density function of the envelope of $n(t)$.
(b) What are the mean and variance of this envelope?

5.31 In the noise analyzer of Figure 5.25a, the low-pass filters are ideal with a bandwidth equal to one-half that of the narrowband noise $n(t)$ applied to the input. Using this scheme, derive the following results:

(a) Equation (5.136), defining the power spectral densities of the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ in terms of the power spectral density of $n(t)$.
(b) Equation (5.137), defining the cross-spectral densities of $n_I(t)$ and $n_Q(t)$.

5.32 Assume that the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the mid-band frequency f_c. Show that the in-phase and quadrature components of $n(t)$ are statistically independent.

5.33

(a) A transmitter at position $x = 0$ emits the signal $A\cos(2\pi f_c t)$. The signal travels at the velocity of light such that a signal at a point on the x-axis is given by

$$r(t,x) = A(x)\cos\left[2\pi f_c\left(t - \frac{x}{c}\right)\right]$$

If the receiver starts at position x_0 and moves at velocity v along the x-axis, what Doppler shift in frequency f_D is observed?

(b) The frequencies of the Doppler shift of the reflected paths in Eq. (5.173) are proportional to the angle of radiation relative to the direction of motion, that is

$$f_n = f_D \cos\psi_n$$

where f_D is maximum Doppler shift. If the multipath angle ψ_n is uniformly distributed over $[-\pi, \pi]$. Compute $\mathbf{E}[\exp(j2\pi f_n \tau)]$. Use this result to prove Eq. (5.174).

5.34 The *modified Bessel function of the first kind zero order* is defined by

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\cos\psi)\, d\psi$$