Solution to Tutorial - 4.

5.4

$$X(t) = \sin(2\pi f t),$$

$$f \sim \text{unif}(0, W).$$

For $X(t)$ to be stationary, it is ~~standard~~ ~~tested~~ necessary that for any distinct time distances t_1 and t_2, the random variables $X(t_1)$ and $X(t_2)$ are identically distributed. (This is a necessary condition, but not a sufficient condition).

We will now show that for

$X(t) = \sin(2\pi f t)$, $f \sim \text{unif}[0, W]$,

$X(t_1)$ and $X(t_2)$ are not identically distributed.

Let $t_1 = 0$ and $t_2 = \frac{1}{16W}$.

It is clear that $X(t_1) = 0$ (probability concentrated at 0) whereas $X(t_2) = \sin(2\pi f t_2)$

$= \sin(2\pi f/_{16\omega})$

Since $f \sim \text{unif}(0, \omega)$.

$X(t_2)$ takes values in

the interval $[0, \sin(\pi/8)]$.

<u>5.7</u>

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

X, Y are i.i.d. $N(0,1)$

(Gaussian zero mean and unit variance)

X & Y are independent of each other.

a) Joint Probability Density Function of (P.D.F.)

$(Z(t_1), Z(t_2))$

Since X & Y are i.i.d. their joint

P.D.F is $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \frac{e^{-(x^2+y^2)/2}}{2\pi}$$

which is the p.d.f of two independent jointly Gaussian random variables.

$\therefore$ X & Y are jointly gaussian distributed.

$\therefore$ For any t ~~t_1 and t_2~~

~~$Z(t_1)$~~ $Z(t) = X \cos 2\pi t + Y \sin 2\pi t$

is a Gaussian random variable.

Also, $Z(t_1)$ and $Z(t_2)$ are Gaussian random variables.

But are they jointly Gaussian?

Indeed Yes, because for any $\lambda_1, \lambda_2 \in \mathbb{R}$

$$V \triangleq \lambda_1 Z(t_1) + \lambda_2 Z(t_2)$$

$$= X(\lambda_1 \cos 2\pi t_1 + \lambda_2 \cos 2\pi t_2)$$

$$+ Y(\lambda_1 \sin 2\pi t_1 + \lambda_2 \sin 2\pi t_2)$$

V is a ~~new~~ Gaussian r.v since X & Y are jointly Gaussian.

Since $Z(t_1)$ and $Z(t_2)$ are jointly Gaussian (as proved above), their joint p.d.f. is given by (let $Z_1 \triangleq Z(t_1)$ and $Z_2 \triangleq Z(t_2)$)

$$f_{Z_1, Z_2}(z_1, z_2) = \frac{1}{(2\pi)\sqrt{K_Z}} e^{-\frac{(z-\mu_z)^T K_Z^{-1}(z-\mu_z)}{2}}$$

where
$$z \triangleq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

$$\mu_z \triangleq \begin{bmatrix} E[Z_1] \\ E[Z_2] \end{bmatrix}, \quad K_Z \triangleq E\left[(z-\mu_z)(z-\mu_z)^T\right].$$

We firstly note that

$$E[Z_1] = E[Z(t_1)] = E[X \cos 2\pi t_1 + Y \sin 2\pi t_1]$$
$$= E[X] \cos 2\pi t_1 + E[Y] \sin 2\pi t_1$$
$$= 0.$$

Similarly $E[Z_2] = E[Z(t_2)] = 0.$

$$\therefore \mu_z = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\therefore \cancel{K_Z = E[\;] = \begin{bmatrix} E(z_1^2) & E(z_1 z_2) \\ E(z_1 z_2) & E(z_2^2) \end{bmatrix}}.$$

Let $Z \triangleq \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, then

$$K_Z = \begin{bmatrix} E[Z_1^2] & E[Z_1 Z_2] \\ E[Z_1 Z_2] & E[Z_2^2] \end{bmatrix}.$$

⑤

$$E(Z_1^2) = E[Z(t_1)^2] = E\left[(X\cos 2\pi t_1 + Y\sin 2\pi t_1)^2\right]$$

$$= E[X^2]\cos^2 2\pi t_1 + E[Y^2]\sin^2 2\pi t_1 + 2E[XY]\sin 2\pi t_1 \cos 2\pi t_1$$

since X, Y are independent

$$E[XY] = E[X]\,E[Y] = 0.$$

$$\therefore E[Z_1^2] = E[X^2]\cos^2 2\pi t_1 + E[Y^2]\sin^2 2\pi t_1$$

$$= \cos^2 2\pi t_1 + \sin^2 2\pi t_1 \quad (\text{since } E[X^2] = E[Y^2] = 1)$$

$$= 1$$

Similarly $E[Z_2^2] = E[Z^2(t_2)] = 1.$

$$E[Z_1 Z_2] = E[Z(t_1)\,Z(t_2)]$$

$$= E\left[(X\cos 2\pi t_1 + Y\sin 2\pi t_1)(X\cos 2\pi t_2 + Y\sin 2\pi t_2)\right]$$

$$= \cos 2\pi t_1 \cos 2\pi t_2 + \sin 2\pi t_1 \sin 2\pi t_2$$

$$= \cos 2\pi(t_1 - t_2).$$

$$\therefore K_2 = \begin{bmatrix} 1 & \cos 2\pi(t_1-t_2) \\ \cos 2\pi(t_1-t_2) & 1 \end{bmatrix} \Rightarrow K_2^{-1} = \frac{1}{|K_2|}\begin{bmatrix} 1 & -\cos 2\pi(t_1-t_2) \\ -\cos 2\pi(t_1-t_2) & 1 \end{bmatrix}$$

where $|K_2| = 1 - \cos^2 2\pi(t_1-t_2)$

$$= \sin^2 2\pi(t_1-t_2).$$

we have

$$K_2^{-1} = \frac{1}{\sin^2 2\pi(t_1-t_2)} \begin{bmatrix} 1 & -\cos 2\pi(t_1-t_2) \\ -\cos 2\pi(t_1-t_2) & 1 \end{bmatrix}$$

$$\therefore \frac{z^T K_2^{-1} z}{2} = \frac{1}{2\sin^2 2\pi(t_1-t_2)} \times$$

$$[z_1 \;\; z_2] \begin{bmatrix} 1 & -\cos 2\pi(t_1-t_2) \\ -\cos 2\pi(t_1-t_2) & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \frac{z_1^2 + z_2^2 - 2 z_1 z_2 \cos 2\pi(t_1-t_2)}{2 \sin^2 2\pi(t_1-t_2)}.$$

$$\therefore f_{Z(t_1),Z(t_2)}(z_1, z_2) = \frac{1}{2\pi \cdot \sin 2\pi(t_1-t_2)} e^{-\frac{(z_1^2+z_2^2-2z_1z_2\cos 2\pi(t_1-t_2))}{2\sin^2 2\pi(t_1-t_2)}}$$

1) Is the process $Z(t)$ stationary? (7).
why?

~~From the previous~~ From the previous Excercise we know that $Z(t)$ is w.s.s., since the auto ~~covariance~~ correlation depends only on the time difference $(t_1 - t_2)$.

If we can show that $Z(t)$ is a Gaussian random process, then ~~it~~ since it is also w.s.s., from property 3 (section 5.9 of the textbook) it will immediately follow that $Z(t)$ is also stationary.

$\therefore$ In the following we show that $Z(t)$ is a Gaussian random process.

To show that $Z(t)$ is Gaussian, (8)
for any $g(t)$, we must show that
$$\int_{-\infty}^{\infty} Z(t)\, g(t)\, dt$$ is a Gaussian r.v.

$$V \triangleq \int_{-\infty}^{\infty} Z(t)\, g(t)\, dt = \int_{-\infty}^{\infty} (X \cos \omega t + Y \sin \omega t)\, g(t)\, dt$$

i.e
$$V = X \left(\int g(t) \cos \omega t \, dt \right) + Y \left(\int g(t) \sin \omega t \, dt \right)$$

$$E[V^2] = \left(\int g(t) \cos \omega t \, dt \right)^2 + \left(\int g(t) \sin \omega t \, dt \right)^2$$
$$= \left| \int g(t)\, e^{j\omega t} \, dt \right|^2$$

$\therefore$ 2nd moment of V is finite for all $g(t)$ ~~which are~~ whose Fourier transform exists, i.e for whom
$$\left| \int_{-\infty}^{\infty} g(t)\, e^{j\omega t} \, dt \right| < \infty .$$

For all such $g(t)$, (Fourier Transformable)

$$V = X \int g(t) \cos 2\pi t \, dt + Y \int g(t) \sin 2\pi t \, dt$$

is a Gaussian r.v. since

X & Y are jointly Gaussian.

This proves that $Z(t)$ is a Gaussian Random Process,

and since in part (a) we had shown that it is w.s.s., from property 3 (Chapter 5.9 of the text book) it follows that $Z(t)$ is a stationary Gaussian random process.

5.16

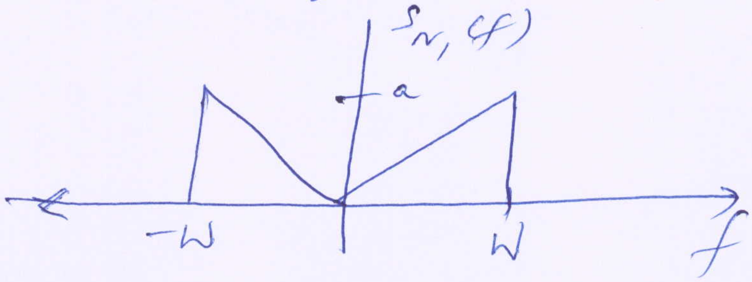
$$n_2(t) = n_1(t)\left(\cos(2\pi f_c t+\theta) - \sin(2\pi f_c t+\theta)\right)$$

f_c is constant,

$\theta \sim \text{unif}\,(-\pi, \pi)$.

i.e.
$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise.} \end{cases}$$

$n_1(t)$ is stationary, with p.s.d.



We need to find the power spectral density (PSD) of $n_2(t)$.

The approach to this problem is

i) Find $R_{n_1}(\tau)$ (inverse fourier transform of $S_{n_1}(f)$)

ii) Find $R_{n_2}(\tau)$

iii) From $R_{n_2}(\tau)$ find $S_{n_2}(f)$ using

$$S_{n_2}(f) = \int_{-\infty}^{\infty} R_{n_2}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

(11)

Step i).

$$R_{n_1}(\tau) = \int S_{n_1}(f)\, e^{j2\pi f\tau}\, df.$$

$$= \int_{-W}^{W} \frac{a}{W} |f|\, e^{j2\pi f\tau}\, df.$$

Here we have used the fact that.

$$S_{n_1}(f) = \begin{cases} \frac{a}{W} |f|, & |f| < W \\ 0, & \text{otherwise.} \end{cases}$$

$$\therefore R_{n_1}(\tau) = \int_{-W}^{0} \frac{-af}{W} e^{j2\pi f\tau}\, df + \int_{0}^{W} \frac{af}{W} e^{j2\pi f\tau}\, df.$$

$$= \int_{0}^{W} \frac{af}{W} e^{-j2\pi f\tau}\, df + \int_{0}^{W} \frac{af}{W} e^{j2\pi f\tau}\, df.$$

$$= \frac{2a}{W} \int_{0}^{W} f \cos 2\pi f\tau\, df$$

$$= \frac{2a}{W} \left[\frac{f \sin 2\pi f\tau}{2\pi\tau} - \frac{1}{2\pi\tau} \int \sin 2\pi f\tau\, df \right]_{0}^{W}$$

$$= \frac{2a}{W} \left[\frac{f \sin 2\pi f\tau}{2\pi\tau} + \frac{\cos 2\pi f\tau}{(2\pi\tau)^2} \right]_{0}^{W}$$

$$= \frac{2a}{W} \left[\frac{W \sin 2\pi W\tau}{2\pi\tau} - \frac{2\sin^2(\pi\tau W)}{(2\pi\tau)^2} \right]$$

$\therefore R_{n_1}(\tau) = 2aW\left[\text{sinc}(2W\tau) - \frac{1}{2}\left(\text{sinc}(W\tau)\right)^2\right].$

ii).

$R_{n_2}(\tau) = E\left[n_2(t)\, n_2(t-\tau)\right].$

$= \cancel{E\left[n_1(t)\, n_1(t-\tau)\, (\cos 2\pi f_c t+\theta)\right.}$

Note that

$n_2(t) = n_1(t)\left(\cos(2\pi f_c t+\theta) - \sin(2\pi f_c t+\theta)\right)$

$= n_1(t)\sqrt{2}\left(\cos(2\pi f_c t+\theta)\cos\pi/4 - \sin\pi/4\,\sin(2\pi f_c t+\theta)\right)$

$= \sqrt{2}\, n_1(t)\, \cos(2\pi f_c t+\theta+\pi/4)$

$\therefore R_{n_2}(\tau) = E\left[n_2(t)\, n_2(t-\tau)\right]$

$= 2E\left[n_1(t)\, n_1(t-\tau)\cos(2\pi f_c t+\theta+\pi/4)\cos(2\pi f_c t+\theta+\pi/4 - 2\pi f_c\tau)\right]$

assuming that $n_1(t)$ and θ are independent, we have

$$R_{n_2}(\tau) = \underset{\substack{n_1(t),\\ n_1(t-\tau)}}{E}\left[n_1(t)\, n_1(t-\tau)\right] \underset{\theta}{E}\left[2\cos(2\pi f_c t + \theta + \pi/4)\cos(2\pi f_c t + \theta + \pi/4 - 2\pi f_c \tau)\right]$$

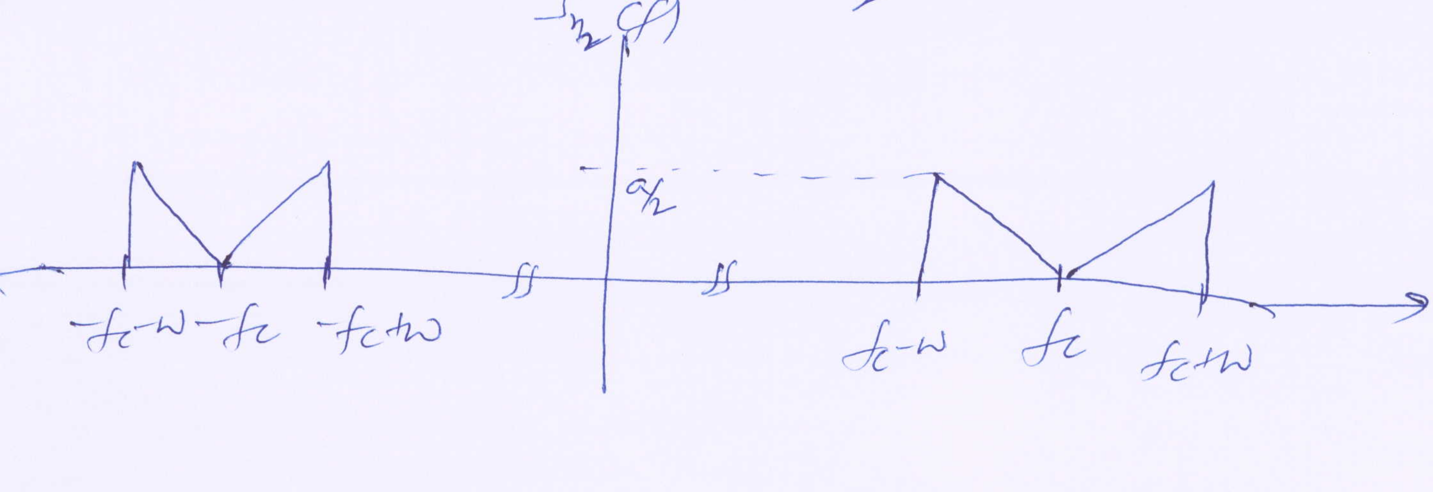
$$= R_{n_1}(\tau)\cos 2\pi f_c \tau \quad (\because n_2(t) \text{ is } \text{w.s.s.})$$

$$= \frac{R_{n_1}(\tau)\, e^{j2\pi f_c \tau}}{2} + \frac{R_{n_1}(\tau)\, e^{-j2\pi f_c \tau}}{2}.$$

iii): taking the ~~inverse~~ Fourier transform of $R_{n_2}(\tau)$ we get

$$S_{n_2}(f) = \int R_{n_2}(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$= \frac{S_{n_1}(f - f_c) + S_{n_1}(f + f_c)}{2}$$



5.19. Since $X(t)$ is a zero mean Gaussian random process,

$X(t_k)$ is a zero mean Gaussian random variable.

Since $X(t)$ is also stationary,

$$E[X^2(t_k)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

$\therefore$ p.d.f of $X(t_k)$ is

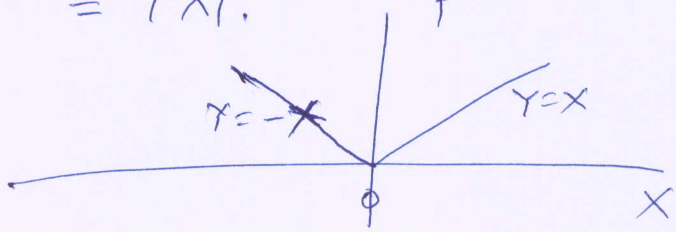
$$f_{X(t_k)}(x) = \frac{1}{\sqrt{2\pi \int_{-\infty}^{\infty} S_X(f)\,df}}\, e^{-\frac{x^2}{2\int_{-\infty}^{\infty} S_X(f)\,df}}.$$

$(-\infty < x < \infty)$.

5.20

We have $X \triangleq X(t_k)$ a Gaussian r.v $N(0, \sigma_X^2)$.

and $Y \triangleq |X|$.



The p.d.f of Y, $f_Y(y) = ??$

The distribution fn of Y, for any $y>0$

$$F_Y(y) = P(Y \le y)$$

$$= P(|X| \le y)$$

$$= P(-y < X \le y)$$

$$= \int_{-y}^{y} f_X(x)\,dx = \int_{-y}^{y} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-x^2}{2\sigma_X^2}}\,dx$$

$$= 1 - 2\int_{y}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-x^2}{2\sigma_X^2}}\,dx.$$

The p.d.f of Y is given by (16)

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - 2\int_y^{\infty} \frac{e^{-x^2/2\sigma_X^2}}{\sqrt{2\pi\sigma_X^2}}\, dx\right)$$

$$= \frac{2}{\sqrt{2\pi\sigma_X^2}}\, e^{-y^2/2\sigma_X^2} \quad , y \geq 0$$

Of course $f_Y(y) = 0$ for $y < 0$.

5.21

$$Y(t) = X^2(t)$$

$X(t)$ is a zero mean, stationary Gaussian process with autocorrelation function $R_X(\tau)$

(a) $E[Y(t)] = E[X^2(t)] = R_X(0)$.

(b) $E[(Y(t) - E(Y(t)))(Y(t-\tau) - E[Y(t-\tau)])]$

$= E[(Y(t) - R_X(0))(Y(t-\tau) - R_X(0))]$

$= E[Y(t)Y(t-\tau)] + R_X^2(0) - 2R_X^2(0)$

$= E[Y(t)Y(t-\tau)] - R_X^2(0)$

$= E[X^2(t)X^2(t-\tau)] - R_X^2(0)$

For computing 17

$$E[X^2(t)X^2(t-\tau)]$$, we use the following famous result for jointly Gaussian r.v.s.

Let A, B, C, D be four jointly Gaussian r.v.'s, then

$$E(ABCD) = E(AB)E(CD) + E(AC)E(BD) + E(AD)E(BC) - 2E(A)E(B)E(C)E(D)$$

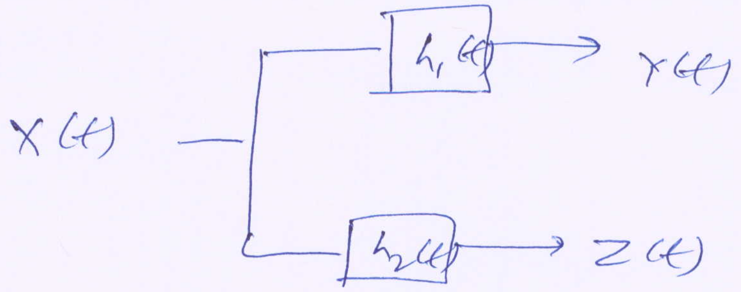
Substituting $A = X(t)$, $B = X(t)$, $C = X(t-\tau)$, $D = X(t-\tau)$, (which are obviously jointly Gaussian), we have

$$\begin{aligned}
E[X^2(t)X^2(t-\tau)] &= E[ABCD] \\
&= E[X^2(t)]\,E[X^2(t-\tau)] + 2\cdot E(X(t)X(t-\tau))\,E(X(t)X(t-\tau)) \\
&= R_X^2(0) + 2R_X^2(\tau)
\end{aligned}$$

$\therefore$ auto covariance $= E[X^2(t)X^2(t-\tau)] - R_X^2(0)$

$$= 2R_X^2(\tau).$$

5.22



Clearly $Y(t)$ and $Z(t)$ are Gaussian random processes

(property 1, section 5.9 of the chapter)

$\therefore$ $Y(t_1)$ and $Z(t_2)$ are Gaussian r.v.s.
They are also jointly Gaussian,

Since for any λ_1, λ_2

$$U \triangleq \lambda_1 Y(t_1) + \lambda_2 Z(t_2)$$

$$= \int \left(\lambda_1 h_1(\tau) X(t_1-\tau) + \lambda_2 h_2(\tau) X(t_2-\tau)\right) d\tau.$$

$$= \int \left(\lambda_1 h_1(t_1-\tau) X(\tau) + \lambda_2 h_2(t_2-\tau) X(\tau)\right) d\tau.$$

$$= \int \underbrace{\left(\lambda_1 h_1(t_1-\tau) + \lambda_2 h_2(t_2-\tau)\right)}_{f(\tau)} X(\tau)\, d\tau.$$

Since $X(t)$ is a Gaussian random process it follows that U is a Gaussian random variable for any chosen λ_1 and λ_2.

$\therefore$ $Y(t_1)$ and $Z(t_2)$ are jointly Gaussian r.v.s.

c) joint p.d.f of $Y(t_1)$, $Z(t_2)$.

$$\mu_1 \triangleq E[Y(t_1)] = E\left[\int_{-\infty}^{\infty} h_1(\tau)\, X(t_1-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} h_1(\tau)\, E(X(t_1-\tau))\,d\tau.$$

$$= \mu_X \int_{-\infty}^{\infty} h_1(\tau)\,d\tau$$

$$= \mu_X H_1(0) \quad \left(H_1(f) = \int_{-\infty}^{\infty} h_1(\tau)\, e^{-j2\pi f\tau}\,d\tau\right)$$

Similarly.

$$\mu_2 \triangleq E[Y(t_2)] = \mu_X H_2(0).$$

$$k_{11} \triangleq E\left[(Y(t_1)-\mu_1)^2\right] = E[Y^2(t_1)] - \mu_1^2$$

$$= \int_{-\infty}^{\infty} S_Y(f)\,df - \mu_1^2$$

$$\therefore \quad k_{11} = \int_{-\infty}^{\infty} S_X(f)\,|H_1(f)|^2\,df - \mu_1^2.$$

where $S_X(f)$ is the p.s.D of $X(t)$.

Similarly.

$$E\left[(Z(t_2)-\mu_2)^2\right] = \int_{-\infty}^{\infty} S_X(f)\,|H_2(f)|^2\,df - \mu_2^2.$$

$$\triangleq k_{22}$$

$$k_{12} \triangleq E\left[(Y(t_1) - \mu_1)(Z(t_2) - \mu_2) \right]$$

$$= E\left[Y(t_1) Z(t_2) \right] - \mu_1 \mu_2$$

$$= E\left[\left(\int_{-\infty}^{\infty} h_1(\tau) X(t_1 - \tau)\, d\tau \right) \left(\int_{-\infty}^{\infty} h_2(\tau) X(t_2 - \tau)\, d\tau \right) \right] - \mu_1 \mu_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2)\, E\left(X(t_1 - \tau_1) X(t_2 - \tau_2) \right) d\tau_1\, d\tau_2 - \mu_1 \mu_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2)\, R_X(t_1 - t_2 + \tau_2 - \tau_1)\, d\tau_1\, d\tau_2 - \mu_1 \mu_2$$

The p.d.f is then given by:

$$\cancel{f_{Y[t_1], Z[t_2]}(y, z) = \frac{1}{(2\pi)|k|} e^{-[y-\mu_1, z] \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}^{-1}}}$$

$$f_{Y[t_1], Z[t_2]}(y, z) = \frac{1}{(2\pi)\sqrt{|k|}}\, e^{-\frac{1}{2} \begin{bmatrix} (y-\mu_1) & (z-\mu_2) \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}^{-1} \begin{bmatrix} y-\mu_1 \\ z-\mu_2 \end{bmatrix}}$$

$$k = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}.$$

5). The necessary and sufficient condition to ensure $Y(t_1)$ and $Z(t_2)$ are statistically independent is that

$$E\left[(Y(t_1)-\mu_1)(Z(t_2)-\mu_2)\right]=0.$$

$$\Longleftrightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)\, h_2(\tau_2)\, R_X(t_1-t_2+\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

$$= \mu_1 \mu_2.$$

$$= \mu_X^2 H_1(0) H_2(0)$$

using the fact that

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{+j2\pi f\tau}\, df$$

we get.

$$\Longleftrightarrow$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)\, h_2(\tau_2) \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(t_1-t_2)}\, e^{j2\pi f\tau_2}\, e^{-j2\pi f\tau_1}\, df\, d\tau_1\, d\tau_2$$

$$= \mu_X^2 H_1(0) H_2(0)$$

(22)

$$\Leftrightarrow \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(t_1-t_2)} \left(\int h_1(\tau_1)\, e^{-j2\pi f\tau_1}\, d\tau_1\right) \left(\int h_2(\tau_2)\, e^{+j2\pi f\tau_2}\, d\tau_2\right) df = \mu_X^2 H_1(0) H_2(0)$$

$$\Leftrightarrow \int_{-\infty}^{\infty} S_X(f)\, H_1(f)\, H_2^*(f)\, e^{j2\pi f(t_1-t_2)}\, df = \mu_X^2 H_1(0) H_2(0).$$

Consider a filter with transfer function

$$A(f) \triangleq S_X(f)\, H_1(f)\, H_2^*(f),$$

and impulse response $a(t)$,

then

$$\int_{-\infty}^{\infty} S_X(f)\, H_1(f)\, H_2^*(f)\, e^{j2\pi f(t_1-t_2)}\, df$$

$$= a(t_1 - t_2) = \mu_X^2 H_1(0) H_2(0)$$

i.e., $Y(t_1)$ and $Z(t_2)$ are independent if and only if

$$a(t_1 - t_2) = \mu_X^2 H_1(0) H_2(0),$$

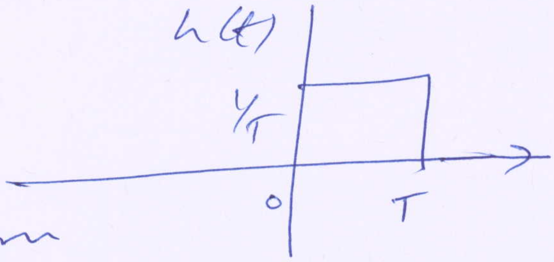
where $a(t)$ is the impulse response of a linear filter with $S_X(f) H_1(f) H_2^*(f)$ as transfer function.

5.23

$$X(t) \longrightarrow \boxed{h(t)} \xrightarrow{t=T} Y.$$

$$\therefore \quad Y = \int_{-\infty}^{T} X(t)\, h(T-t)\, dt.$$

$$= \int_{0}^{T} X(t) \cdot \frac{1}{T}\, dt \quad \text{since}$$



clearly, since $X(t)$ is a Gaussian random process, Y is a Gaussian random variable.

a) mean and variance of Y.

$$E(Y) = E\left[\frac{1}{T}\int_0^T X(t)\,dt\right]$$

$$= 0 \qquad \text{since} \quad E[X(t)] = 0 \text{ for all } t.$$

$$E(Y^2) = E\left[\frac{1}{T^2}\int_0^T\!\!\int_0^T X(t_1)X(t_2)\, dt_1 dt_2\right] \quad (\because X(t) \text{ is stationary})$$

$$= \frac{1}{T^2}\int_0^T\!\!\int_0^T E\left(X(t_1)\,X(t_2)\right) dt_1 dt_2.$$

autocorrelation fn of $X(t)$

$$= \frac{1}{T^2}\int_0^T\!\!\int_0^T R_X(t_1 - t_2)\, dt_1\, dt_2$$

$$= \frac{1}{T^2}\int_0^T\!\!\int_0^T\!\!\int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(t_1 - t_2)}\, dt_1\, dt_2\, df$$

(24)

$$\therefore E(Y^2) = \frac{1}{T^2} \int_0^T \int_0^T \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(t_1 - t_2)}\, dt_1\, dt_2\, df$$

$$= \frac{1}{T^2} \int_{-\infty}^{\infty} S_X(f) \left(\int_0^T e^{j2\pi f t_1}\, dt_1 \right) \left(\int_0^T e^{-j2\pi f t_2}\, dt_2 \right) df$$

$$\int_0^T e^{j2\pi f t}\, dt = \left(\frac{e^{j2\pi f t}}{j2\pi f} \right)_0^T$$

$$= \frac{e^{j2\pi f T} - 1}{j2\pi f}$$

$$= \frac{e^{j\pi f T}}{\pi f} \frac{(e^{j\pi f T} - e^{-j\pi f T})}{2j}$$

$$= e^{j\pi f T} \frac{\sin \pi f T}{\pi f}$$

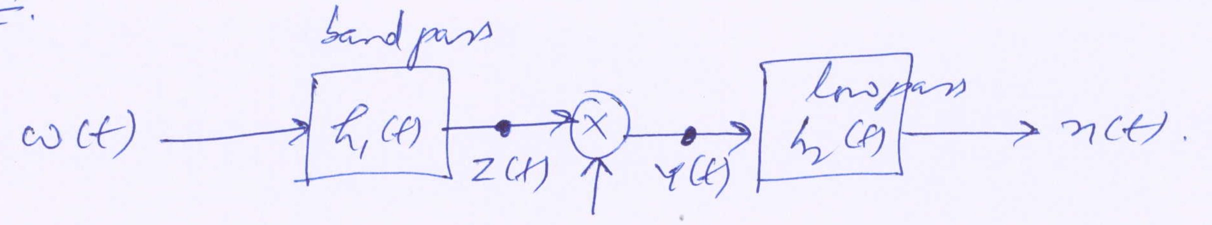
$$= \int_{-\infty}^{\infty} S_X(f) \left| \frac{1}{T} \int_0^T e^{j2\pi f t}\, dt \right|^2 df$$

$$= \int_{-\infty}^{\infty} S_X(f)\, (\operatorname{sinc}(fT))^2\, df.$$

$\therefore$ p.d.f of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi \int_{-\infty}^{\infty} S_X(f) \operatorname{sinc}^2(fT)\, df}}\; e^{-\frac{y^2}{2 \int_{-\infty}^{\infty} S_X(f) \operatorname{sinc}^2(fT)\, df}}$$

5.27



$\cos(2\pi f_c t + \theta)$ where $\theta \sim \text{unif}(-\pi, \pi)$

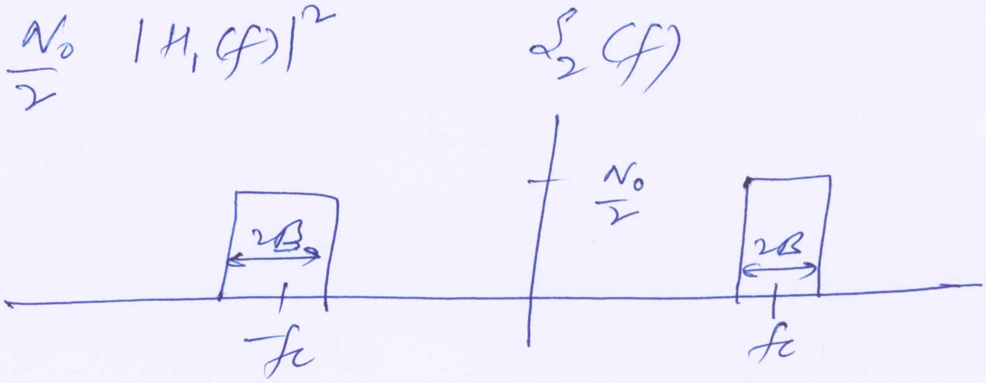
$w(t)$ is ~~colour~~ white gaussian of zero mean and p.s.d. $\frac{N_0}{2}$,

i.e $R_w(\tau) = \frac{N_0}{2}\delta(\tau)$.

$\therefore w(t)$ is stationary (is assumed).

The p.s.d. $S_Z(f)$ of the output of $Z(t)$ the bandpass filter is given by.

$$S_Z(f) = \frac{N_0}{2}|H_1(f)|^2$$



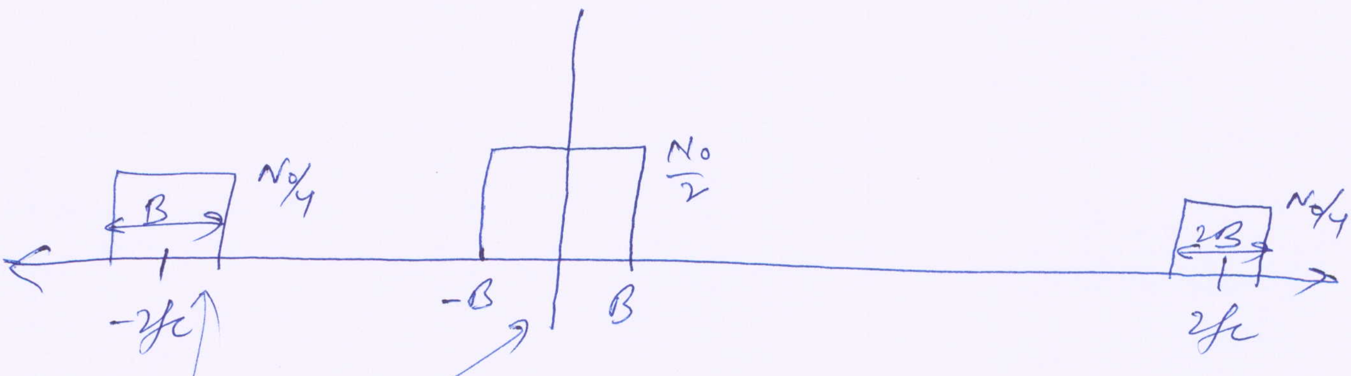
the p.s.d of $Y(t) \triangleq Z(t)\cos(2\pi f_c t + \theta)$

$$= Z(t)\frac{\left(e^{j(2\pi f_c t + \theta)} + e^{-j(2\pi f_c t + \theta)}\right)}{2}$$

is

$$S_Y(f) = \frac{S_Z(f - f_c) + S_Z(f + f_c)}{2}$$

$\therefore$ $S_Y(f)$.



non-overlapping since $-2f_c + B < -B$
as $f_c > B$.

Finally the p.s.d of $n(t)$ is

$$S_N(f) = S_Y(f)\,|H_2(f)|^2$$

$$= \begin{cases} N_0/2, & |f| < B \\ 0, & \text{otherwise.} \end{cases}$$

$$\therefore \quad E[n^2(t)] = R_N(0) = \int_{-\infty}^{\infty} S_N(f)\,df = \int_{-B}^{B} \frac{N_0}{2}\,df$$

$$= N_0 B.$$

and $E[n(t)] = 0$ (since $w(t)$ was zero mean).

$$R_N(\tau) = \int S_N(f) e^{j2\pi f\tau} df = \frac{N_0}{2}\int_{-B}^{B} e^{j2\pi f\tau} df$$

$$= \frac{N_0}{2}\left[\frac{e^{j2\pi f\tau}}{j2\pi\tau}\right]_{-B}^{B} = N_0 B\,\text{sinc}(2B\tau)$$

c) Since $R_N(\tau) = N_0 B \operatorname{Sinc}(2B\tau)$

Clearly $n(t)$ is a Gaussian random process (since $w(t)$ is passed through linear filters).

$\therefore$ $n(t)$ and $n(t-\tau)$ are jointly Gaussian r.v.s. having zero mean and identical distribution.

and $E[n(t)\, n(t-\tau)]$

$= R_N(\tau) = N_0 B \operatorname{Sinc}(2B\tau)$

if $n(t)$ is sampled at every T seconds then the samples (for any integer k) $n(t)$ and $n(t+kT)$ have autocorrelation

$$E[n(t)\, n(t+kT)] = R_N(kT)$$
$$= N_0 B \operatorname{Sinc} 2BkT$$

if $T = \frac{1}{2B}$, then we note that

$E[n(t)\, n(t+kT)] = 0$ for any $k \neq 0$.

$\therefore$ Samples are uncorrelated (and hence independent) if ~~sample~~ $n(t)$ is sampled at $2B$ samples per second.